Visual Algebra

Lecture 2.9: Semidirect products, intuitively

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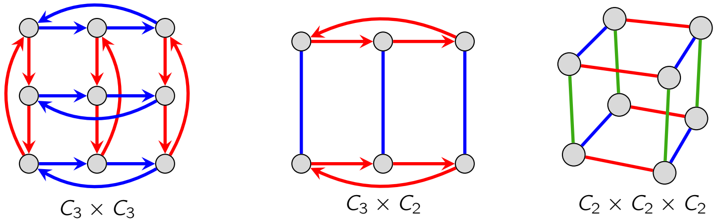
Revisiting direct products

Let A, B be groups with identity elements 1_A and 1_B. Suppose we have a

- Cayley graph of A with generators $a_1, \dots, a_k$,
- Cayley graph of B with generators $b_1, \dots, b_\ell$.

We can create a Cayley graph for $A \times B$, by taking

- **Vertex set**: $\{(a, b) \mid a \in A, \; b \in B\}$,
- **Generators**: $(a_1, 1_B), \dots, (a_k, 1_B)$ and $(1_A, b_1), \dots, (1_A, b_\ell)$.



Remark

"A-arrows" are independent of "B-arrows." Algebraically, this means

$$(a, 1_B) * (1_A, b) = (a, b) = (1_A, b) * (a, 1_B).$$

Revisiting direct products

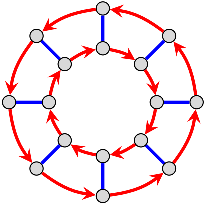
Remark

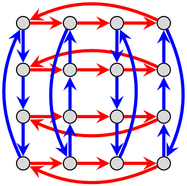
Just because a group is not written with $\times$ does not mean that there is not secretly a direct product structure lurking behind the scenes.

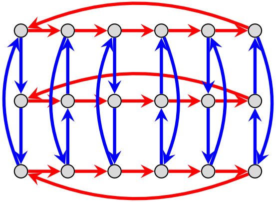
We have already seen that $V_4 \cong C_2 \times C_2$, and that $C_6 \cong C_3 \times C_2$.

However, sometimes it is even less obvious.

Two of the following three groups secretly have a direct product structure.







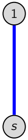
(And it's probably not the two you think.)

The "inflation method" for constructing direct products

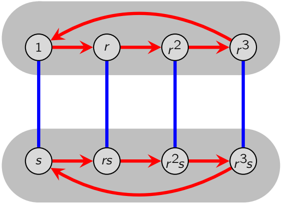
Semidirect products are a more general construction than the direct product.

They can be thought of as a "twisted" version of the direct product.

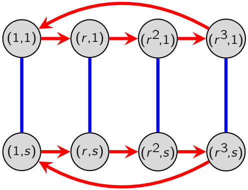
To motivate this, consider the following "inflation method" for constructing the Cayley graph of a direct product:



Start with a copy of $B = C_2$



Inflate each node, insert $A = C_4$ in each and connect corresponding nodes with edges



"pop" each inflated node to get the direct product $C_4 \times C_2$

Consider this process, but with the red arrows reversed in the bottom inflated node.

This would result in a Cayley graph for the group D_4.

We say that D_4 is the semidirect product of C_4 and C_2, written $D_4 \cong C_4 \rtimes C_2$.

Rewirings of Cayley graphs

Reversing the red arrows worked is because it was a structure-preserving rewiring.

Formally, this is an automorphism, which is an isomorphism from a group to itself.

We'll learn more about this when we study homomorphisms. Just know that it's a bijection

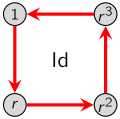
$$\varphi \colon G \longrightarrow G$$

satisfying some extra properties.

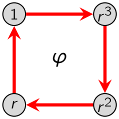
There are two ways to describe a rewiring:

- fix the position of the nodes and rewire the edges
- fix the position of the edges and relabel the nodes.

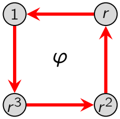
This is best seen with an example:



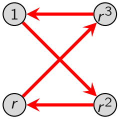
Cayley graph of C_4



edges rewired



nodes relabeled



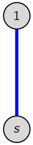
non-rewiring

The graph on the right isn't allowed because it doesn't preserve the algebraic structure.

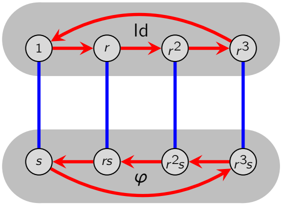
The "inflation method" for constructing semidirect products

Semidirect products can be constructed via the "inflation process" for $A \times B$, but *insert φ-rewired copies of the Cayley graph for A into inflated nodes of B.*

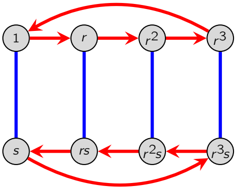
Let's construct $A \rtimes B$ for $A = C_4$ and $B = C_2$, with the rewiring φ from the previous slide.



Start with a copy of $B = C_2$



Inflate each node, insert rewired versions of $A = C_4$, and connect corresponding nodes



"pop" each inflated node to get the semidirect product $C_4 \rtimes_\varphi C_2 \cong D_4$

In the middle graph, each inflated node of $B = C_2 = \langle s \rangle$ is labeled with a re-wiring.

Formally, this is a just map

$$\theta \colon C_2 \longrightarrow \mathbf{Aut}(C_4), \qquad \theta(g) = \begin{cases} \mathrm{Id} & g = 1 \\ \varphi & g = s, \end{cases}$$

where $\theta(g)$ specifies which re-wiring gets put into the inflated node g of C_2.

Semidirect products

There are strong restrictions for inserting rewirings of the Cayley graph of A into B.

The map θ must be a structure-preserving map, called a homomorphism.

If we stick a φ-rewiring into the inflated node $b \in B$, then we must insert a φ^2-rewiring into node $b^2 \in B$, and so on.

Definition (informal)

Consider groups A, B, and a structure-preserving map

$$\theta \colon B \longrightarrow \mathrm{Aut}(A)$$

to the set of rewirings of A. The **semidirect product** $A \rtimes_\theta B$, is constructed by:

- inflating the nodes of the Cayley graph of B, [*mnemonic*: B for "ballooon"]
- inserting a $\theta(b)$-rewiring of the Cayley graph A into node b of B,
- For each edge bewteen B-nodes, connect corresponding pairs of A-nodes with that edge.

Semidirect products

Key point

For groups A, B and map

$$\theta\colon B \longrightarrow \mathrm{Aut}(A),$$

the image $\theta(b)$ can be thought of as "*which rewiring node $b \in B$ gets label with*".

Any group A always has a trivial rewiring.

Remark

For the trivial map $\theta\colon B \longrightarrow \mathrm{Aut}(A)$ sending everything to the identity rewiring

$$A \rtimes_\theta B = A \times B.$$

For any n, there is a rewiring φ of $C_n = \langle r \rangle$ that "reverses all of the r-arrows".

The semidirect product of C_n and $C_2 = \{1, s\}$, with respect to

$$\theta\colon C_2 \longrightarrow \mathrm{Aut}(C_n), \qquad \theta(g) = \begin{cases} \mathrm{Id} & g = 1 \\ \varphi & g = s, \end{cases}$$

is $D_n \cong C_n \rtimes_\theta C_2$.

Semidirect products

Reasons for introducing semidirect products this early

- it helps us understand a new way to construct groups
- it helps us understand the structure of some groups we've already seen
- thinking about *what* works in this process and *why*, helps us gain a more holistic understanding about group theory
- it will be easier to learn advanced concepts such as automorphisms if we get a preview of them in advance, and gain intutition

Proposition

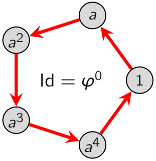
The set of rewirings of a Cayley graph of G forms a group, denoted $\mathbf{Aut}(G)$.

Moreover, this group does not depend on the Cayley graph, but on the group itself.

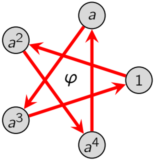
Rewirings and the automorphism group

There are four rewirings (i.e., automorphisms) of the Cayley graph of $C_5 = \langle a \rangle$.

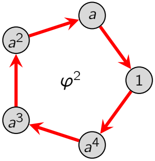
Every rewiring can be realized by iterating the "doubling map" $\varphi\colon C_5 \to C_5$ that replaces each instance of a with a^2, i.e., a length-k path with a length-$2k$ path.



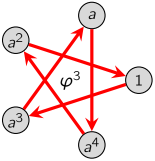
starting graph



$a^1 \mapsto (a^1)^2 = a^2$



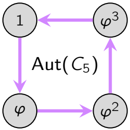
$a^2 \mapsto (a^2)^2 = a^4$

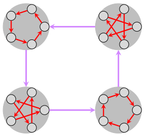


$a^4 \mapsto (a^4)^2 = a^3$

Notice that the rewirings form a group:

$$\mathrm{Aut}(C_5) = \{1,\ \varphi,\ \varphi^2,\ \varphi^3\} \cong C_4$$





Remark

For any group G, the set $\mathrm{Aut}(G)$ of rewirings forms a group, called its automorphism group.

The automorphism group of C_n

Each automorphism is defined by where it sends a generator: $r \mapsto r^k$.

"*each red arrow gets multiplied by k*"

The group $\text{Aut}(C_n)$ is isomorphic to the group with operation multiplication modulo n:

$$U_n := \{k \mid 0 < k < n,\ \gcd(n,k) = 1\}.$$

Example:

$$\text{Aut}(C_7) \cong U_7 = \{1,2,3,4,5,6\} = \langle 3 \rangle \cong C_6$$

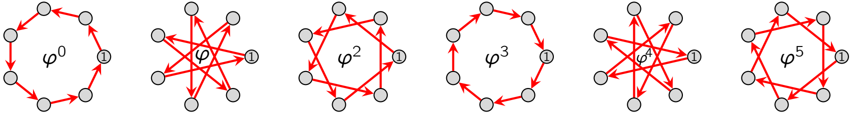
$$2^0 = 1, \quad 2^1 = 2, \quad 2^2 = 4, \quad 2^3 = 1$$

$$3^0 = 1, \quad 3^1 = 3, \quad 3^2 = 2$$

$$3^3 = 6, \quad 3^4 = 4, \quad 3^5 = 5$$

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

Since $U_7 = \langle 3 \rangle$, the re-wirings of C_7 are generated by the "*tripling map*" $r \stackrel{\varphi}{\longmapsto} r^3$.



$C_7 = \langle r \rangle$ $\quad r^1 \mapsto (r^1)^3 = r^3$ $\quad r^3 \mapsto (r^3)^3 = r^2$ $\quad r^2 \mapsto (r^2)^3 = r^6$ $\quad r^6 \mapsto (r^6)^3 = r^4$ $\quad r^4 \mapsto (r^4)^3 = r^5$

An example: the automorphism group of C_7