# Visual Algebra

## Lecture 2.10: Examples of semidirect products

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# Quick recap

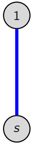
## Definition (informal)

Consider groups $A, B$, and a structure-preserving map

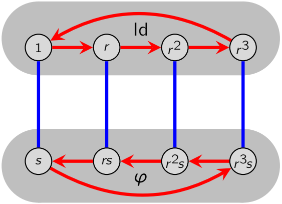
$$\theta \colon B \longrightarrow \mathrm{Aut}(A)$$

to the group of rewirings of $A$. The **semidirect product** $A \rtimes_\theta B$, is constructed by:

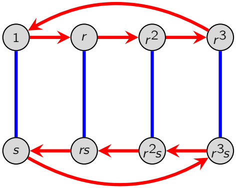
- inflating the nodes of the Cayley graph of $B$, [*mnemonic*: $B$ for "balloon"]
- inserting a $\theta(b)$-rewiring of the Cayley graph of $A$ into node $b$ of $B$,
- For each edge bewteen $B$-nodes, connect corresponding pairs of $A$-nodes.




Start with a copy of $B = C_2$




Inflate each node, insert rewired versions of $A = C_4$, and connect corresponding nodes

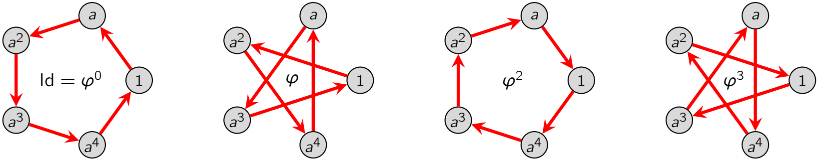



"pop" each inflated node to get the semidirect product $C_4 \rtimes_\varphi C_2 \cong D_4$

# One more recap from the last video

There are four rewirings (i.e., automorphisms) of the Cayley graph of $C_5 = \langle a \rangle$.

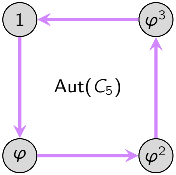
Every rewiring can be realized by iterating the "doubling map" $\varphi \colon C_5 \to C_5$ that replaces each instance of $a$ with $a^2$, i.e., a length-$k$ path with a length-$2k$ path.

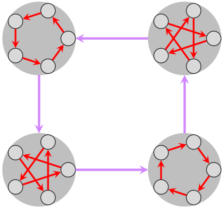



*starting graph* $\qquad a^1 \mapsto (a^1)^2 = a^2 \qquad a^2 \mapsto (a^2)^2 = a^4 \qquad a^4 \mapsto (a^4)^2 = a^3$

The rewirings form the automorphism group:

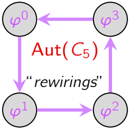
$$\mathrm{Aut}(C_5) = \{1, \varphi, \varphi^2, \varphi^3\} \cong C_4$$

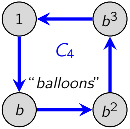


# The $1^{\text{st}}$ semidirect product of $C_5$ and $C_4$

Let's construct a semidirect product $C_5 \rtimes_{\theta_1} C_4$:

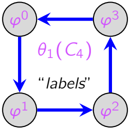


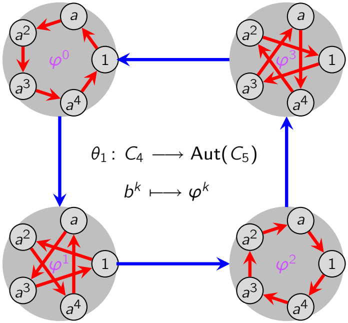

"*labeling map*"

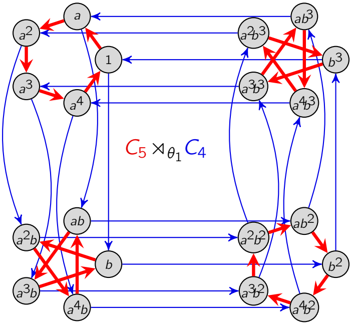
$C_4 \xrightarrow{\theta_1} \mathrm{Aut}(C_5)$

$b^k \longmapsto \varphi^k$



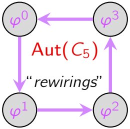

Stick in rewired copies of $A$, and then reconnect the $B$-arrows.

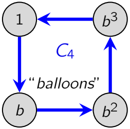


# The 2nd semidirect product of $C_5$ and $C_4$

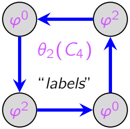
Let's now construct a different semidirect product, $C_5 \rtimes_{\theta_2} C_4$:



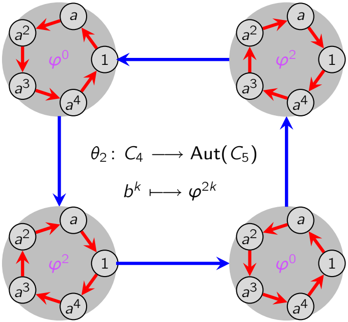



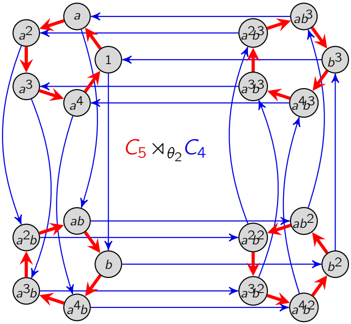

"*labeling map*"

$$C_4 \xrightarrow{\theta_2} \mathrm{Aut}(C_5)$$
$$b^k \longmapsto \varphi^{2k}$$



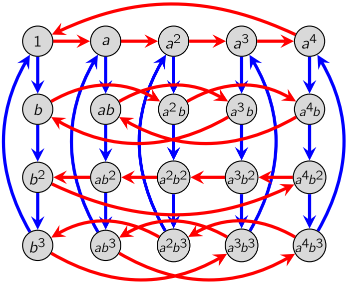

Stick in rewired copies of $A$, and then reconnect the $B$-arrows.

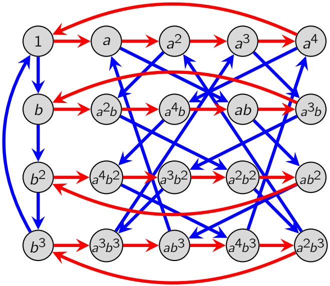


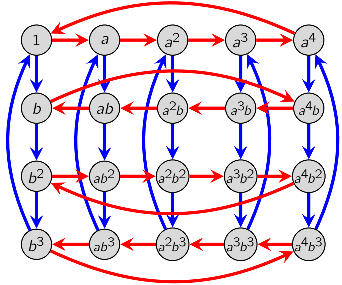
# Rewiring edges vs. relabeling nodes

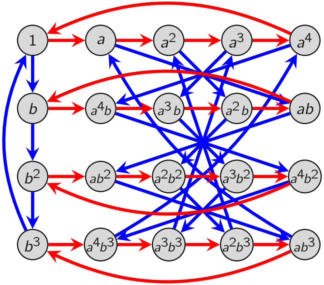
$C_5 \rtimes_{\theta_1} C_4$



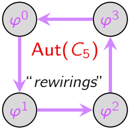

$C_5 \rtimes_{\theta_2} C_4$

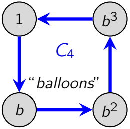


# The 3rd semidirect product of $C_5$ and $C_4$

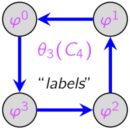
Let's construct another semidirect product $C_5 \rtimes_{\theta_3} C_4$:



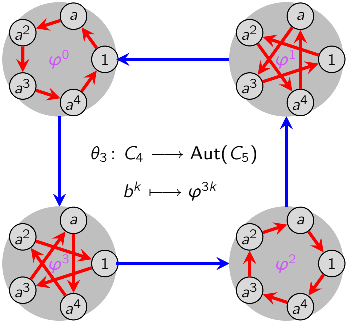



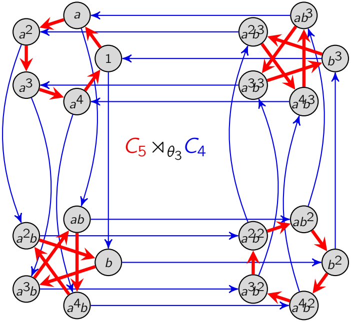

"*labeling map*"

$$C_4 \xrightarrow{\theta_3} \mathrm{Aut}(C_5)$$
$$b^k \longmapsto \varphi^{3k}$$



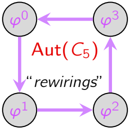

Stick in rewired copies of $A$, and then reconnect the $B$-arrows.

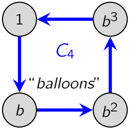


# The direct product of $C_5$ and $C_4$

Let's now construct the "trivial" semidirect product, $C_5 \rtimes_{\theta_0} C_4 = C_5 \times C_4$:

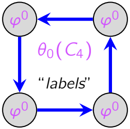


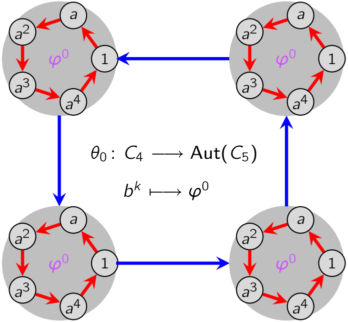

"*labeling map*"

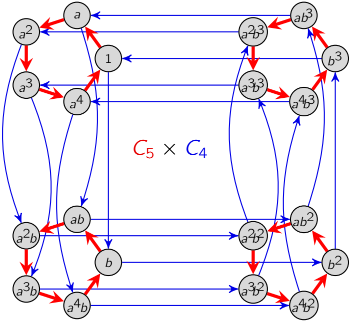
$C_4 \xrightarrow{\theta_0} \mathrm{Aut}(C_5)$

$b^k \longmapsto \varphi^0$




Stick in rewired copies of $A$, and then reconnect the $B$-arrows.

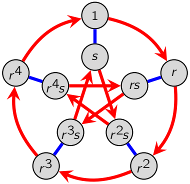


# Semidirect products

## Questions

- does our semidirect product construction actually yield a group?
- (what would happen if we try $C_5$ and $C_2$?)
- when do 2 labeling maps give isomorphic semidirect products?
- is the semidirect product commutative?




*not a group*

*Which groups did we encounter when constructing $C_5 \rtimes_{\theta_k} C_4$, for $k = 1, 2, 3$?*

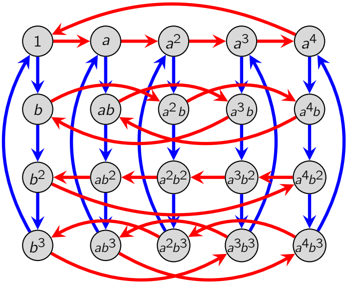
It turns out that there are only three nonabelian groups of order 20:

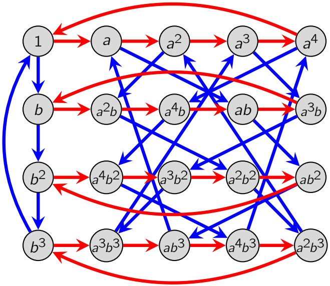
1. the dihedral group $D_{10}$
2. the dicyclic group $\mathrm{Dic}_{10}$
3. a 1D "*affine group*" $\mathrm{AGL}_1(\mathbb{Z}_5) \cong \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} : a, b \in \mathbb{Z}_5,\ a \neq 0 \right\} \leq \mathrm{GL}_2(\mathbb{Z}_5)$.

We'll answer these questions and more later, when we study automorphisms.

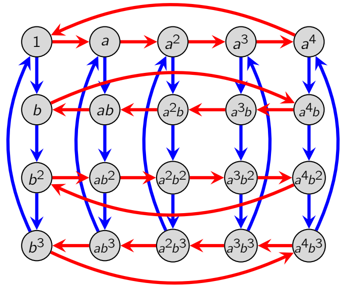
# What are the orders of the elements in these groups?

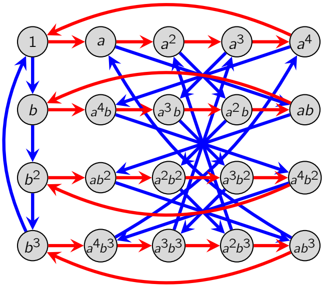
$C_5 \rtimes_{\theta_1} C_4$



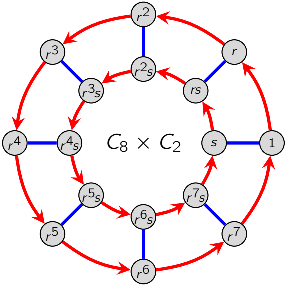



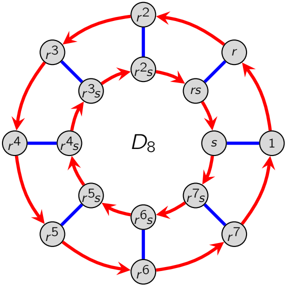

$C_5 \rtimes_{\theta_2} C_4$

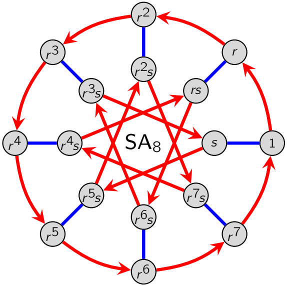


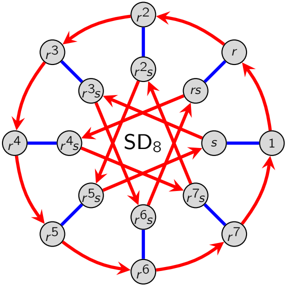



# Four groups we've seen of order 16



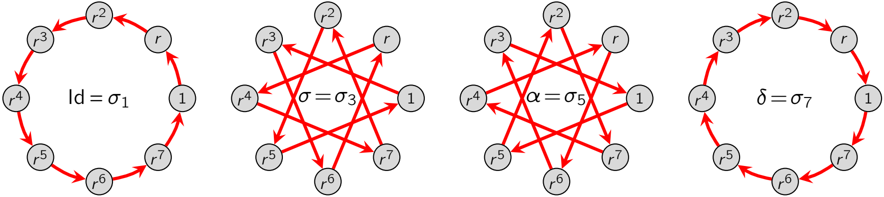



# Semidirect products of $C_8$ and $C_2$

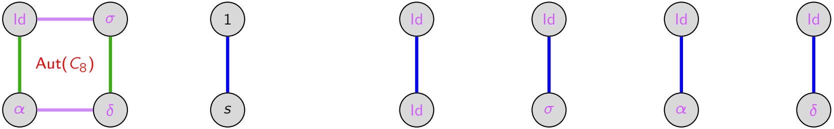
There are four rewirings of the Cayley graph $C_8 = \langle r \rangle$:




All three non-trivial rewirings have order 2:

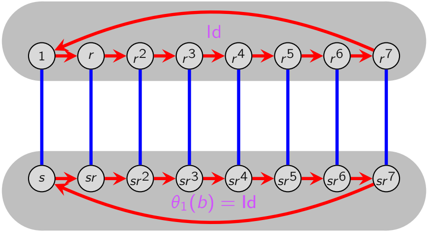
$$r \xrightarrow{\sigma} r^3 \xrightarrow{\sigma} (r^3)^3 = r^9 = r, \qquad r \xrightarrow{\alpha} r^5 \xrightarrow{\alpha} (r^5)^5 = r^{25} = r, \qquad r \xrightarrow{\delta} r^7 \xrightarrow{\delta} (r^7)^7 = r^{49} = r.$$

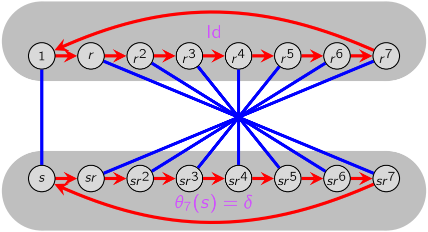
There are four labeling maps $\theta_k \colon C_2 \longrightarrow \mathrm{Aut}(C_8) \cong V_4$:

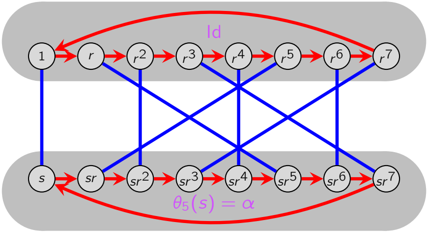


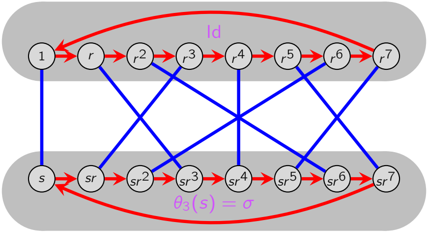

$s \xmapsto{\theta_1} \mathrm{Id}$ $\quad s \xmapsto{\theta_3} \sigma$ $\quad s \xmapsto{\theta_5} \alpha$ $\quad s \xmapsto{\theta_7} \delta$

# The four semidirect products $C_8 \rtimes_i C_2$



# Semidirect products of $C_{2^m}$ and $C_2$

## Theorem

For each $n = 2^m$, there are four distinct semidirect products of $C_n$ with $C_2$:

1. $C_n \rtimes_{\theta_1} C_2 \cong C_n \times C_2$,
2. $C_n \rtimes_{\theta_\alpha} C_2 \cong \mathrm{SA}_n$,
3. $C_n \rtimes_{\theta_\delta} C_2 \cong D_n$,
4. $C_n \rtimes_{\theta_\sigma} C_2 \cong \mathrm{SD}_n$,

where the rewirings are maps $C_{2^m} \longrightarrow C_{2^m}$ defined by

$$r \overset{\theta_1}{\longmapsto} r, \qquad r \overset{\theta_\sigma}{\longmapsto} r^{2^{m-1}-1}, \qquad r \overset{\theta_\alpha}{\longmapsto} r^{2^{m-1}+1}, \qquad r \overset{\theta_\delta}{\longmapsto} r^{-1}.$$

The reason why this holds is that $\theta(b)$ in $\mathrm{Aut}(C_{2^m})$ must be an order of order 1 or 2, because $\theta(b^2) = \theta(1) = \mathrm{Id}$.

There are only three elements of order 2 in the group $U_{2^m}$, due to the following result from number theory.

## Lemma

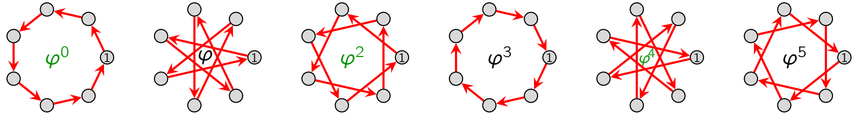
For any $n \geq 3$, the quadratic equation

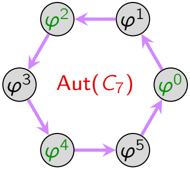
$$x^2 \equiv 1 \pmod{2^n}$$

has exactly four distinct solutions, $\pm 1$ and $2^{n-1} \pm 1$.

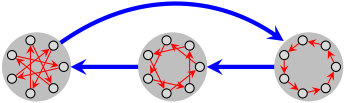
# The smallest nonabelian group of odd order: $C_7 \rtimes_\theta C_3$

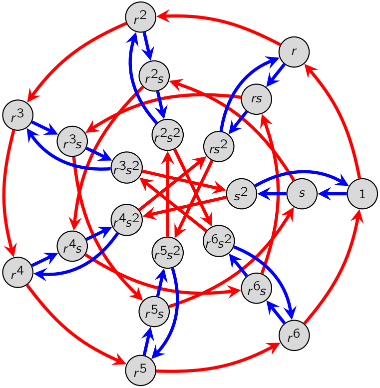
There are 6 rewirings (automorphisms) of $C_7$:




$$C_3 \xrightarrow{\ \theta\ } \mathrm{Aut}(C_7)$$
$$s^k \longmapsto \varphi^{2k}$$



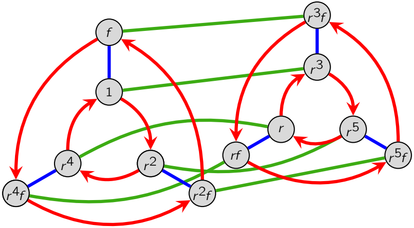


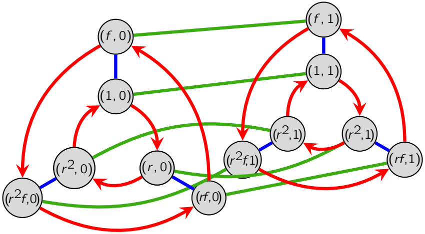
# A surprising fact

We know that we can construct the dihedral group $D_6$ as a semidirect product $C_6 \rtimes_\theta C_2$.

But it also secretly decomposes as a *direct product*!

To see this, let's draw a Cayley graph with a nonstandard generating set, $D_6 = \langle r^2, r^3, f \rangle$.




It is apparent that $D_6 \cong D_3 \times \mathbb{Z}_2 = \langle (r,0), (f,0), (0,1) \rangle$!

**Question**: *How does this generalize to larger dihedral groups?*

We'll understand this better later when we study subgroups and automorphisms.