Visual Algebra

Lecture 2.12: Other finite groups

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Fundamental "building block" groups

The complete classification of finite groups is an impossible task.

However, work along these lines is worthwhile, because much can be learned from studying the structure of groups.

Open-ended question

What group structural properties are possible, what are impossible, and how does this depend on $|G|$?

One approach is to first understand basic "building block groups," and then deduce properties of larger groups from these building blocks, and how to put them together.

In chemistry, "building blocks" are atoms. In number theory, they are prime numbers.

What is a group theoretic analogue of this?

There are several possible answers.

One approach is to study groups that cannot be collapsed by a nontrivial quotient. These are called simple.

The classification of finite simple groups was completed in 2004. It took over 10000 pages of mathematics spread over 500 papers and 50+ years.

p-groups

A different approach to classify groups is motivated by the following:

to understand groups of order $72 = 2^3 \cdot 3^2$, it would be helpful to first understand groups of order $2^3 = 8$ and $3^2 = 9$.

Definition

If p is prime, then a p-group is any group G of order p^n.

Let's look at small powers of p.

Every group of order p is cyclic, and hence abelian. We can ask:

For what other integers n do there not exist any nonabelian groups?

We don't yet have the tools to answer this. But let's investigate for small powers of p:

Groups of order p^2.

- There are only two: $\mathbb{Z}_{p^2}$ and $\mathbb{Z}_p \times \mathbb{Z}_p$.

Groups of order p^3. Staring with $p = 2$:

- three are abelian: $\mathbb{Z}_{p^3}$, $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$, and $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$
- the dihedral group D_4
- the quaternion group Q_8.

Theorem

For each prime p, there are 5 groups of order p^3.

Surprisingly, the pattern for $p = 2$ does not generalize.

Groups of order p^3, for $p > 2$

- the Heisenberg group over $\mathbb{Z}_p$,

$$\text{Heis}(\mathbb{Z}_p) := \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} : a, b, c \in \mathbb{Z}_p \right\} \cong C_p^2 \rtimes C_p,$$

- another group defined as

$$G_p := \left\{ \begin{bmatrix} 1 + pm & b \\ 0 & 1 \end{bmatrix} : m, b \in \mathbb{Z}_{p^2} \right\} \cong C_{p^2} \rtimes C_p.$$

These generalize from p^3 to p^{1+2n}, and are called extraspecial p-groups:

$$M(p) = \langle a, b, c \mid a^p = b^p = c^p = (ab)^2 = (ac)^2 = 1,\ ab = abc \rangle,$$
$$N(p) = \langle a, b, c \mid a^p = b^p = c,\ (ab)^2 = (ac)^2 = 1,\ ab = abc \rangle.$$

Groups of order ≤ 30

order	groups	order	groups	order	groups	order	groups
1	C_1	12 (cont.)	A_4	18 (cont.)	$D_3 \times C_3$	24 (cont.)	$Q_8 \times C_3$
2	C_2	13	C_{13}		$C_3 \rtimes D_3$		$D_3 \times C_4$
3	C_3	14	C_{14}	19	C_{19}		$D_3 \times C_2^2$
4	C_4		D_7	20	C_{20}		$C_3 \rtimes C_8$
	C_2^2	15	C_{15}		$C_{10} \times C_2$		$C_3 \rtimes D_4$
5	C_5	16	C_{16}		D_{10}		C_{25}
6	C_6		$C_8 \times C_2$		Dic_{10}		$C_5 \times C_5$
	D_3		C_4^2		$\mathrm{AGL}_1(\mathbb{Z}_5)$	26	C_{26}
7	C_7		$C_4 \times C_2^2$	21	C_{21}		D_{13}
8	C_8		C_2^4		$C_7 \rtimes C_3$	27	C_{27}
	$C_4 \times C_2$		D_8	22	C_{22}		$C_9 \times C_3$
	C_2^3		SD_8		D_{22}		C_3^3
	D_4		SA_8	23	C_{23}		$C_9 \rtimes C_3$
	Q_8		Q_{16}	24	C_{24}		$C_3^2 \rtimes C_3$
9	C_9		$D_4 \times C_2$		$C_{12} \times C_2$	28	C_{28}
	$C_3 \times C_3$		$Q_8 \times C_2$		$C_6 \times C_2^2$		$C_{14} \times C_2$
10	C_{10}		$C_4 \rtimes C_4$		D_{12}		D_{14}
	$C_5 \times C_2$		$C_2^2 \rtimes C_4$		Dic_{12}		Dic_{14}
11	C_{11}		DQ_8		S_4	29	C_{29}
12	C_{12}	17	C_{17}		$\mathrm{SL}_2(\mathbb{Z}_3)$	30	C_{30}
	$C_6 \times C_2$	18	C_{18}		$A_4 \times C_2$		D_{15}
	D_6		$C_6 \times C_3$		$\mathrm{Dic}_{12} \times C_2$		$D_5 \times C_3$
	Dic_6		D_9		$D_4 \times C_3$		$D_3 \times C_5$

The online LMFDB (https://lmfdb.org/Groups/Abstract/)

⌂ → Groups → Abstract

Abstract groups

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The database currently contains 544,831 groups from many different sources, the largest of which is S_{47} of order $47!$. In addition, it contains 275,379,753 of their subgroups and 39,933,457 of their irreducible complex characters. You can browse further statistics.

Learn more

- Source and acknowledgements
- Completeness of the data
- Reliability of the data
- Abstract group labeling

Browse

By order: 1-64 65-127 128 129-255 256 257-383 384 385-511 513-1000 1001-1500 1501-2000 2001-

By nilpotency class: 1 2 3 4 5 6 7 8 9 (and not nilpotent)

By property: abelian nonabelian solvable nonsolvable simple perfect rational

Some interesting groups or a random group

Search for subgroups or complex characters

Search

Advanced search options

Order	3	e.g. 4, or a range like 3..5	Exponent	2, 3, 7	e.g. 2, or list of integers like 2, 3, 7
Automorphism group	4.2	e.g. 4.2	Nilpotency class	3	e.g. 4, or a range like 3..5
Automorphism group order	3	e.g. 4, or a range like 3..5	Commutator	4.2, 8	e.g. 4 or 4.2 (order or label)
Center	4.2, 8	e.g. 4 or 4.2 (order or label)	Abelianization	4.2, 8	e.g. 4 or 4.2 (order or label)
Central quotient	4.2, 8	e.g. 4 or 4.2 (order or label)			
Abelian			Direct product		
Cyclic			Semidirect product		
Nilpotent			Perfect		
Simple			Solvable		
Transitive degree	3	e.g. 4, or a range like 3..5	Permutation degree	3	e.g. 4, or a range like 3..5
Number of subgroups	3	e.g. 4, or a range like 3..5	Number of normal subgroups	3	e.g. 4, or a range like 3..5
Number of conjugacy classes	3	e.g. 4, or a range like 3..5			
Order statistics	1^1, 2^3, 3^2	e.g. 1^1, 2^3, 3^2			
Results to display	50				

Display: List of groups | Random group

The number of groups of order n is...

1009. 1
1010. 6
1011. 2
1012. 13
1013. 1
1014. 23
1015. 2
1016. 12
1017. 2
1018. 2
1019. 1
1020. 37
1021. 1
1022. 4
1023. 2
1024. 49,487,367,289

The number of p-groups, for $p = 2, 3, 5$ is...

2.	1	3.	1	5.	1
4.	2	9.	2	25.	2
8.	5	27.	5	125.	5
16.	14	81.	15	625.	15
32.	51	243.	67	3125.	77
64.	267	729.	504	15625.	684
128.	2,328	2187.	9,310	78125.	34,297
256.	56,092	6561.	unknown	390625.	unknown
512.	10,494,213				
1024.	49,487,367,289				
2048.	$> 1.774 \times 10^{15}$				

"The human race will never know the exact number of groups of order 2048.*"*
–John Conway (Princeton University)

Almost all finite groups are 2-groups

Amazing fact

There are 49,910,533,149 groups of order $|G| \leq 2000$.

Of these, 49,487,367,289 of them (99.2%!) have order 1024

Only 6 groups of order 1024 have an element of order 512:

- C_{1024}
- Q_{1024}.
- $C_{512} \times C_2$
- D_{512}
- SA_{512}
- SD_{512}

Conjecture

Almost all finite groups are 2-groups. That is,

$$\lim_{n\to\infty} \frac{\#\text{ 2-groups groups of order} \leq n}{\#\text{ of groups of order} \leq n} = 1.$$

Fun resources for exploring finite groups

- The **GroupNames** website (comprehensive list):

  https://people.maths.bris.ac.uk/~matyd/GroupNames/

- **LMFDB**: database of L-functions, modular forms, and related objects

  https://beta.lmfdb.org/Groups/Abstract/

- The interactive **GroupExplorer** website (only small groups):

  https://nathancarter.github.io/group-explorer/index.html

- The free open source **GAP** (Groups, Algorithms, Programming) software package:

  https://www.gap-system.org/

  and a nice Mac interface called **Gap.app**:

  https://cocoagap.sourceforge.io/