Visual Algebra

Lecture 3.2: Subgroup lattices

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Recall from last time

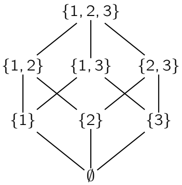
A lattice is a partially ordered set such that every pair of elements x, y has a unique:

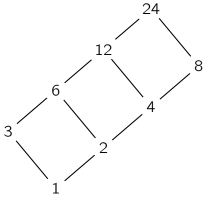
- supremum, or least upper bound, $x \vee y$
- infimum, or greatest lower bound, $x \wedge y$.

Examples that we're familiar with are subset lattices and divisor lattices.

$x \vee y = x \cup y$

$x \wedge y = x \cap y$

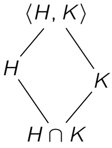




$x \vee y = \text{lcm}(x, y)$

$x \wedge y = \text{gcd}(x, y)$

In this lecture, we'll see lots of examples of subgroup lattices.



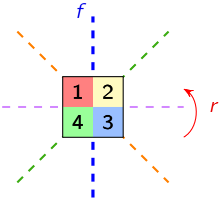
$H \vee K$: "*smallest subgroup above both H and K*"

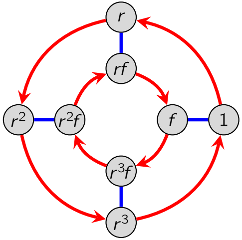
$H \wedge K$: "*largest subgroup below both H and K*"

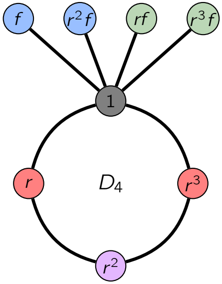
The subgroup lattice of D_4

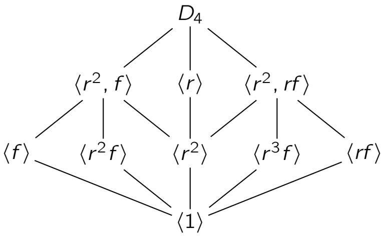
The subgroups of D_4 are:

- The entire group D_4, and the trivial group $\langle 1 \rangle$
- 4 subgroups generated by reflections: $\langle f \rangle$, $\langle rf \rangle$, $\langle r^2 f \rangle$, $\langle r^3 f \rangle$
- 1 subgroup generated by a 180° rotation, $\langle r^2 \rangle \cong C_2$
- 1 subgroup generated by a 90° rotation, $\langle r \rangle \cong C_4$
- 2 subgroups isomorphic to V_4: $\langle r^2, f \rangle$, $\langle r^2, rf \rangle$.

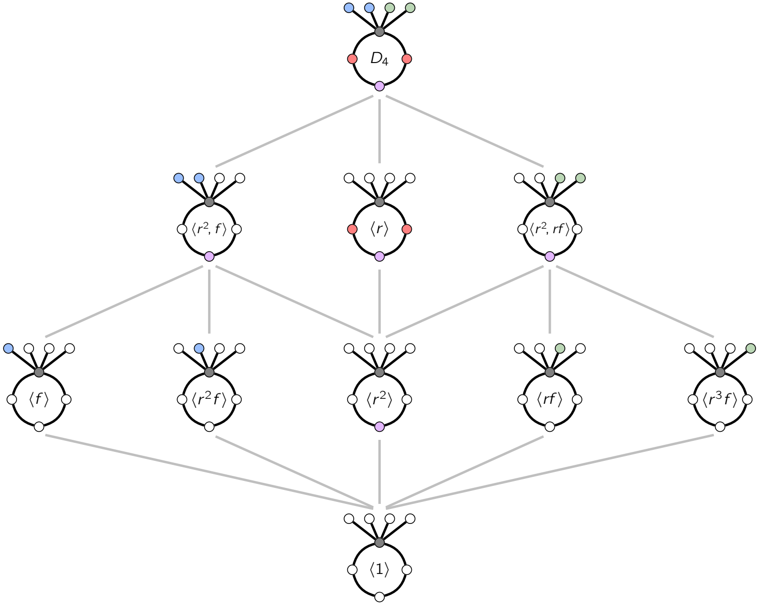




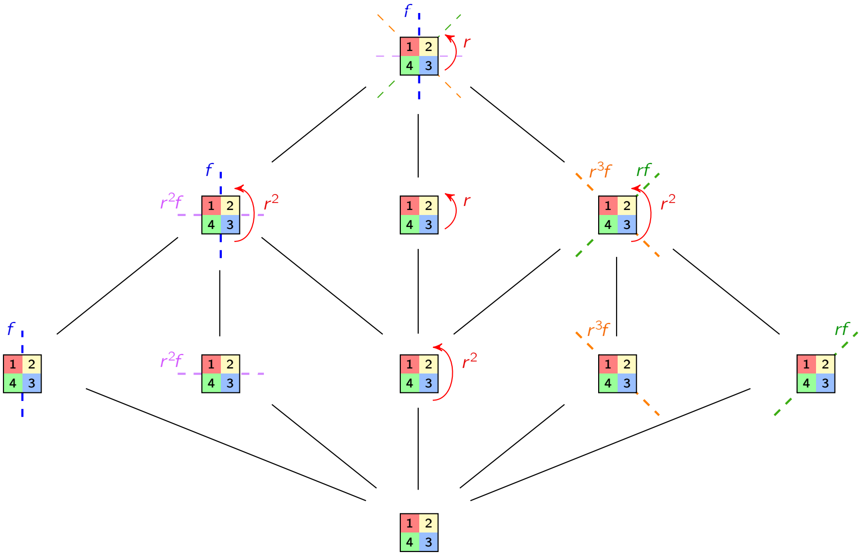




The subgroup lattice of D_4



The subgroup lattice of D_4



The subgroup lattice of Q_8

Let's determine all subgroups of the quaternion group

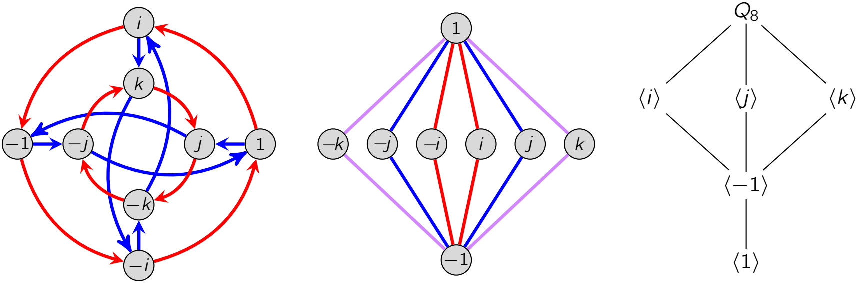
$$Q_8 = \langle i, j, k \mid i^2 = j^2 = k^2 = ijk = -1 \rangle.$$

Every element generates a cyclic subgroup:

$$\langle 1 \rangle = \{1\}, \qquad \langle -1 \rangle = \{\pm 1\}, \qquad \langle i \rangle = \langle -i \rangle = \{\pm 1, \pm i\},$$

$$\langle j \rangle = \langle -j \rangle = \{\pm 1, \pm j\}, \qquad \langle k \rangle = \langle -k \rangle = \{\pm 1, \pm k\}.$$

There are no other proper subgroups.

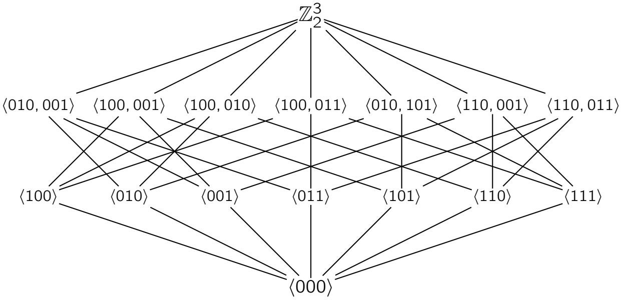


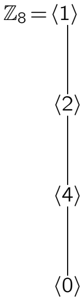
The subgroup lattices of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_8$

All $\binom{7}{2} = 21$ pairs of non-identity elements generate a subgroup isomorphic to V_4.

But this triple-counts all such subgroups. In summary, the subgroups of $\mathbb{Z}_2^3$ are:

- The subgroups G and $\{000\}$,
- 7 subgroups isomorphic to C_2,
- 7 subgroups isomorphic to V_4.





Subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

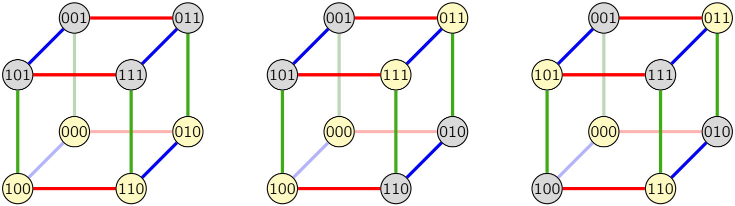
We've seen the subgroup lattices of four groups of order 8:

- D_4 has five elements of order 2, and 10 subgroups.
- Q_8 has one element of order 2, and 6 subgroups.
- $\mathbb{Z}_2^3$ has seven *elements* of order 2, and 16 subgroups.

Rule of thumb

Groups with elements of small order tend to have more subgroups than those with elements of large order.

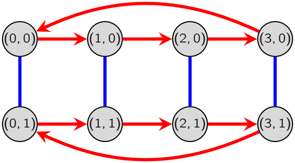
The following Cayley graphs show three different subgroups of order 4 in $\mathbb{Z}_2^3$.

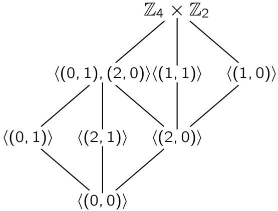


$\langle 100, 010 \rangle$ $\langle 100, 011 \rangle$ $\langle 110, 011 \rangle$

Groups of order 8

There is one more group of order 8 whose subgroup lattice we have not yet seen.





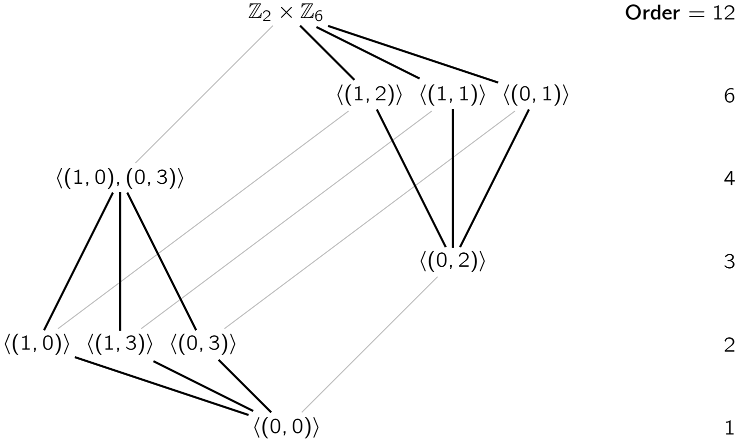
Let's summarize the sizes of the subgroups of the groups of order 8 that we have seen.

	C_8	Q_8	$C_4 \times C_2$	D_4	C_2^3
# elts. of order 8	4	0	0	0	0
# elts. of order 4	2	6	4	2	0
# elts. of order 2	1	1	3	5	7
# elts. of order 1	1	1	1	1	1
# subgroups	4	6	8	10	16

Observations

- Groups that have more elements of small order tend to have more subgroups.
- In all of these cases, the order of each subgroup divides $|G|$.

Examples of subgroup lattices



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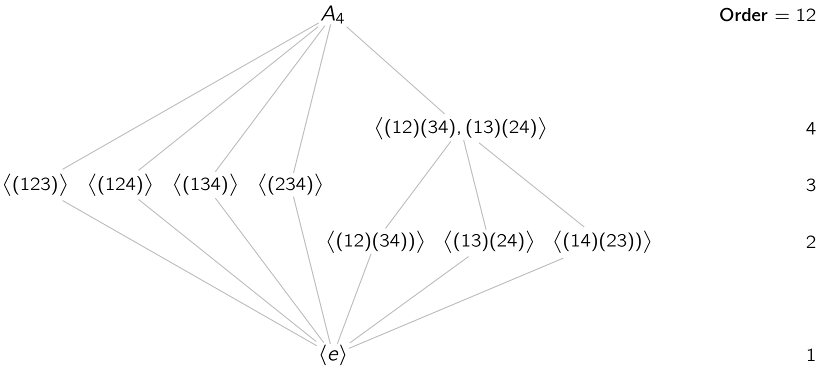

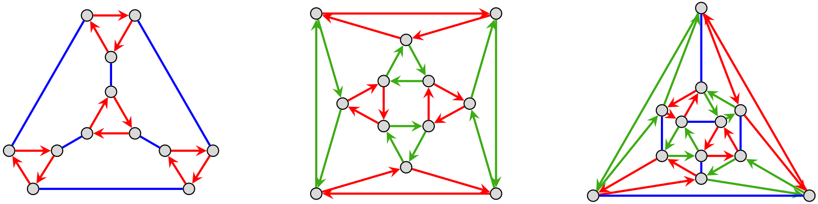

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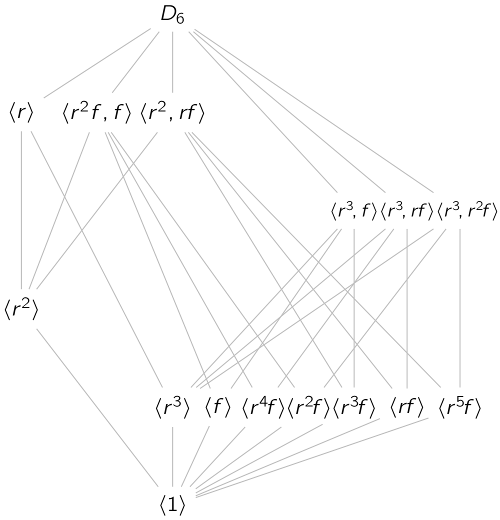

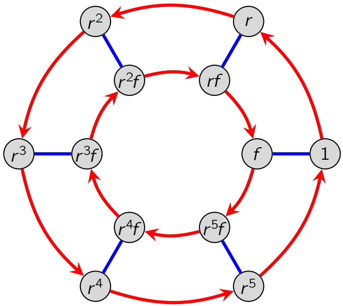

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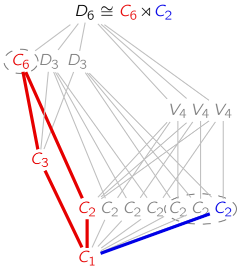
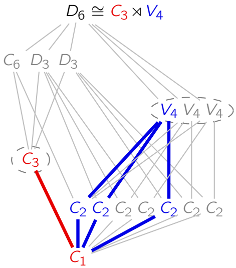
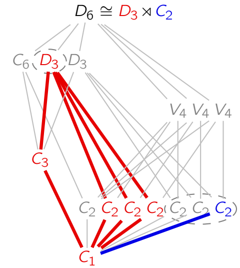
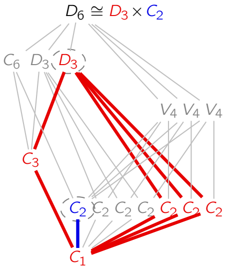


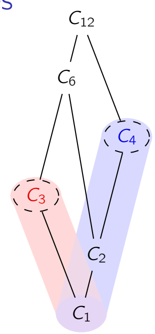


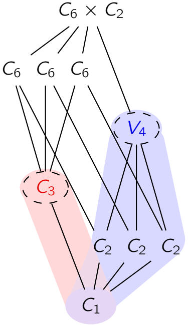


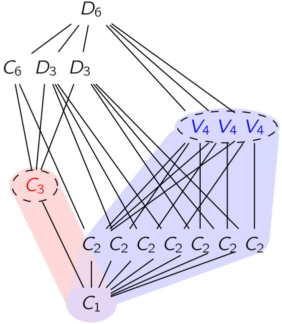


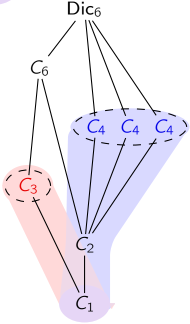
Examples of subgroup lattices

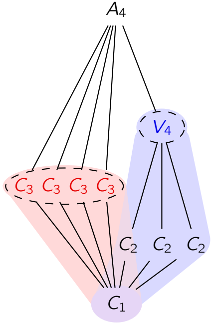
There are five groups of order 12.



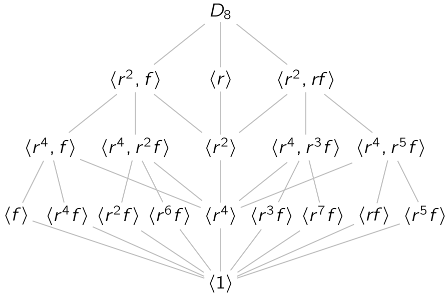


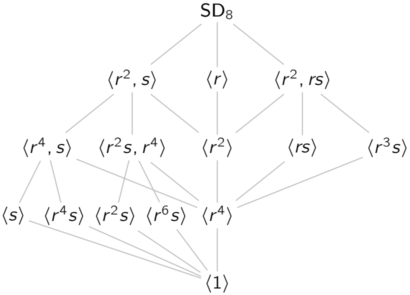


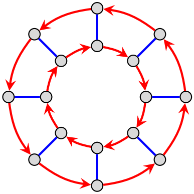


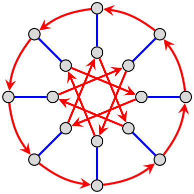


Examples of subgroup lattices









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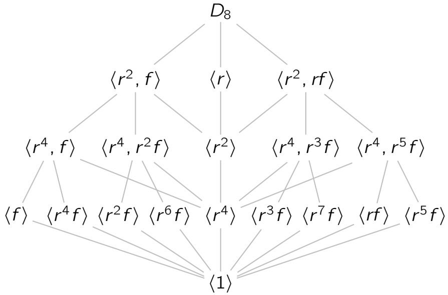

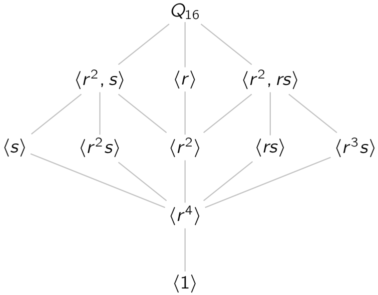

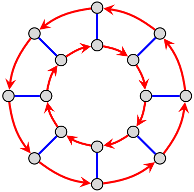

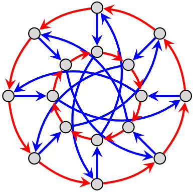

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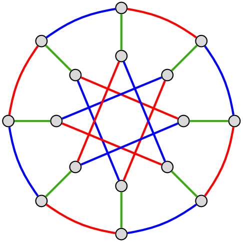

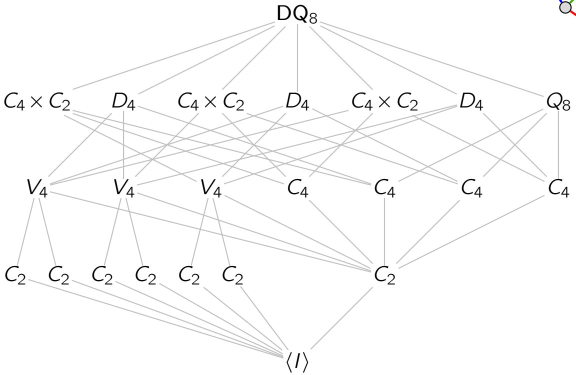

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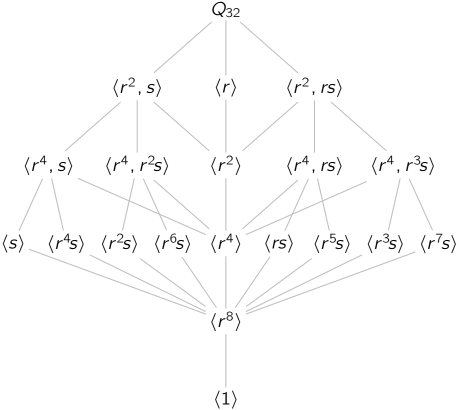

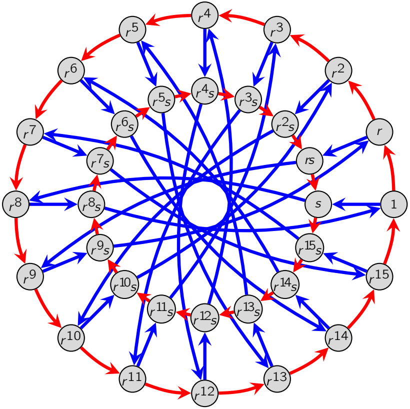

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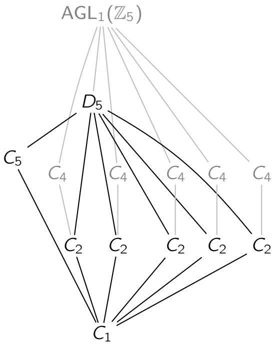
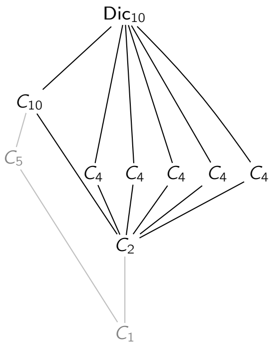




Order $= 20$

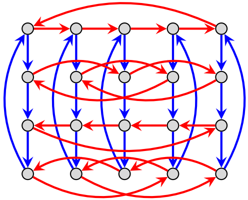
10

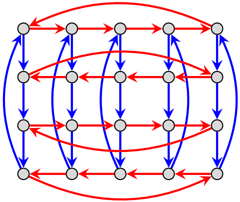
5

4

2

1





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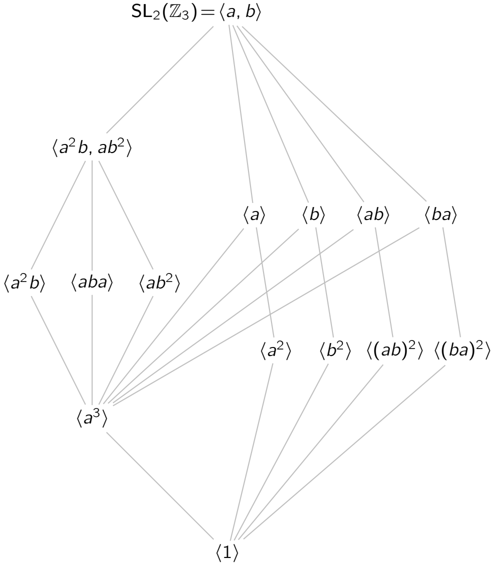

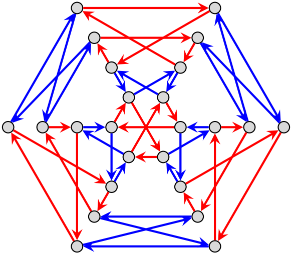

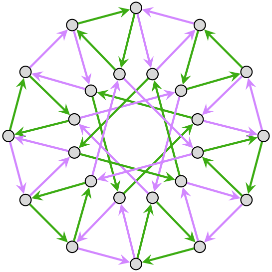

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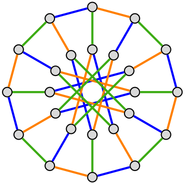

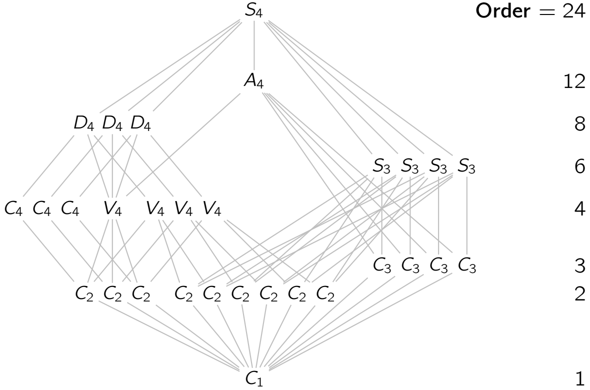

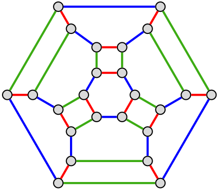

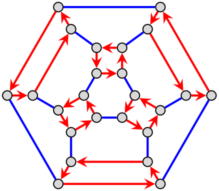

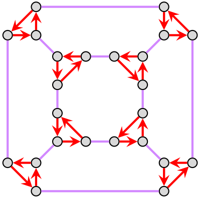

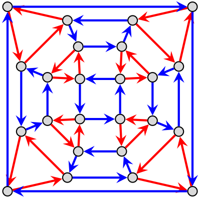

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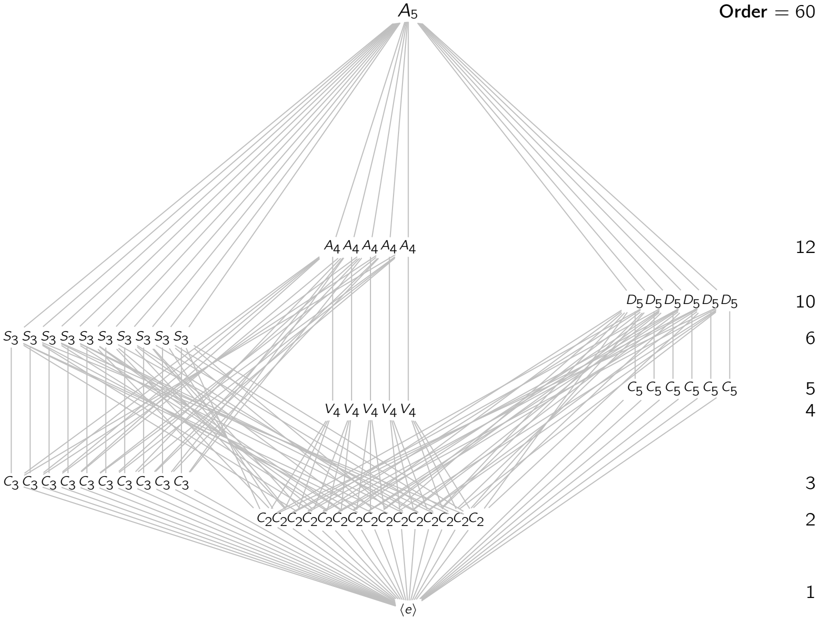

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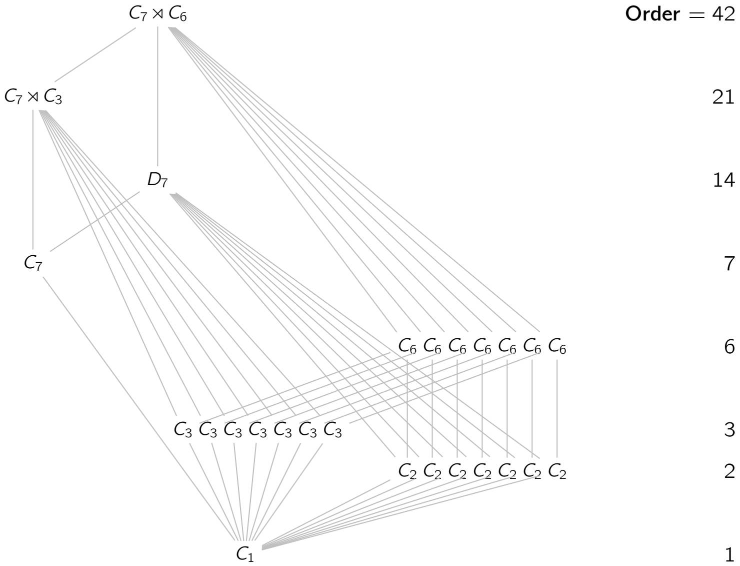

Examples of subgroup lattices