Visual Algebra

Lecture 3.3: Cosets

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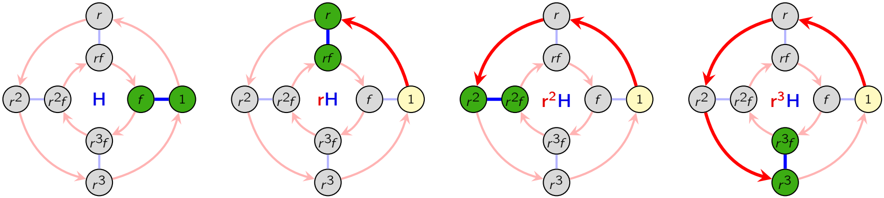
The idea of cosets

By the regularity property of Cayley graphs, identical copies of the fragment that corresponds to a subgroup appears throughout the graph.



Of course, only one of these is actually a subgroup; the others don't contain the identity.

These are called left cosets of $H = \langle f \rangle$.

Informal definition

To find the left coset xH in a Cayley graph, carry out the the following steps:

1. starting from the identity, follow a path to get to x ("*follow the x-path*")
2. from x, follow all "H-paths".

Cosets, formally

Definition

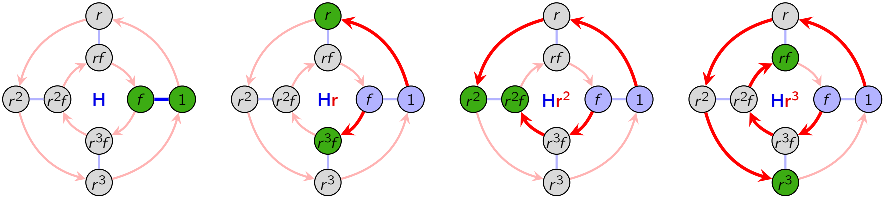
If $H \leq G$, then a left coset is a set

$$xH = \{xh \mid h \in H\},$$

for some fixed $x \in G$ called the representative. Similarly, we can define a right coset as

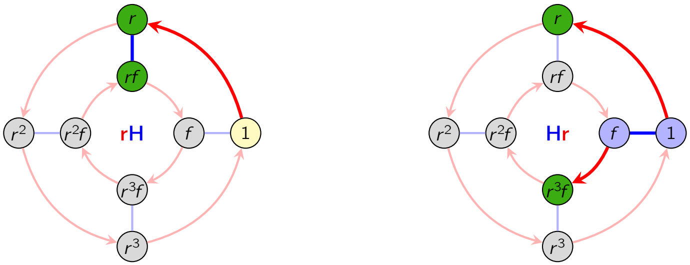
$$Hx = \{hx \mid h \in H\}.$$

Let's look at the right cosets of $H = \langle f \rangle$ in D_4.



Left vs. right cosets

- The **left coset** rH in D_4: first traverse the r-path, then traverse all "H-paths".
- The **right coset** Hr in D_4: first traverse all H-paths, then traverse the r-path.



$rH = r\{1, f\} = \{r, rf\} = rf\{f, 1\} = rfH$ $\qquad$ $Hr = \{1, f\}r = \{r, r^3f\} = \{f, 1\}r^3f = Hr^3f$

Left cosets look like copies of the subgroup. Right cosets are usually scattered, because we adopted the convention that arrows in a Cayley graph represent right multiplication.

Key point

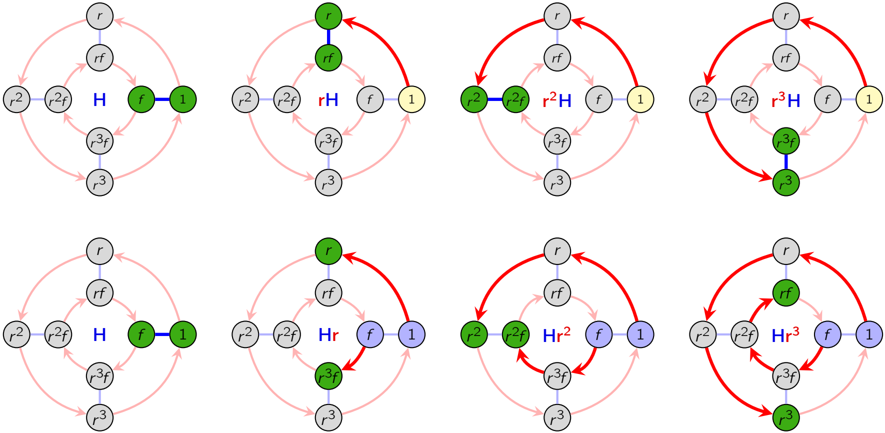
Left and right cosets are generally different.

Left vs. right cosets

Definition

Let $H \leq G$. Given $x \in G$, its left coset xH and right coset Hx are:

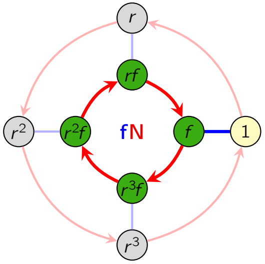
$$xH = \{xh \mid h \in H\}, \qquad Hx = \{hx \mid h \in H\}.$$



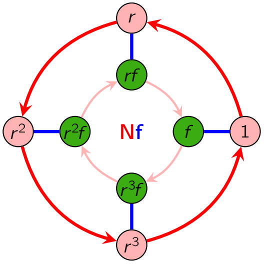
Left vs. right cosets

Let's look at the left and right cosets of a different subgroup, $N = \langle r \rangle$.

- The **left coset** fN in D_4: first traferse the f-path, then traverse all "N-paths".
- The **right coset** Nf in D_4: first traverse all N-paths, then traverse the f-path.



$fN = f\{1, r, r^2, r^3\} = \{f, fr, fr^2, fr^3\}$



$Nf = \{1, r, r^2, r^3\}f = \{f, rf, r^2f, r^3f\}$

Remarks

- There are multiple representatives for the same coset:

$$fN = rfN = r^2fN = r^3fN, \qquad Nf = Nrf = Nr^2f = Nr^3f.$$

- For this subgroup, each left coset is a right coset. Such a subgroup is called normal.

Basic properties of cosets

The following results should be "visually clear" from the Cayley graphs and regularity.

Proposition

Each (left) coset can have multiple representatives: if $b \in aH$, then $aH = bH$.

Proof

Since $b \in aH$, we can write $b = ah$, for some $h \in H$. That is, $h = a^{-1}b$ and $a = bh^{-1}$.

To show that $aH = bH$, we need to verify both $aH \subseteq bH$ and $aH \supseteq bH$.

"$\subseteq$": Take $ah_1 \in aH$. We need to write it as bh_2, for some $h_2 \in H$. By substitution,

$$ah_1 = (bh^{-1})h_1 = b(h^{-1}h_1) \in bH.$$

"$\supseteq$": Pick $bh_3 \in bH$. We need to write it as ah_4 for some $h_4 \in H$. By substitution,

$$bh_3 = (ah)h_3 = a(hh_3) \in aH.$$

Therefore, $aH = bH$, as claimed. □

Corollary (boring but useful)

The equality $xH = H$ holds if and only if $x \in H$. (And analogously, for $Hx = H$.)

Basic properties of cosets

Proposition

For any subgroup $H \leq G$, the (left) cosets of H partition the group G.

Proof

We know that the element $g \in G$ lies in a (left) coset of H, namely gH. Uniqueness follows because if $g \in kH$, then $gH = kH$. □

Proposition

All (left) cosets of $H \leq G$ have the same size. □

Proof

It suffices to show that $|xH| = |H|$, for any $x \in H$.

Define a map

$$\phi \colon H \longrightarrow xH, \qquad h \longmapsto xh.$$

It is elementary to show that this is a bijection. □

Lagrange's theorem

Remark

For any subgroup $H \leq G$, the left cosets of H partition G into subsets of equal size.

The right cosets also partition G into subsets of equal size, but *they may be different.*

Let's compare these two partitions for the subgroup $H = \langle f \rangle$ of $G = D_4$.

H	$\mathbf{r^2H}$	**rH**	$\mathbf{r^3H}$
f	$r^2 f$	rf	r^3
1	r^2	r	$r^3 f$

H	$\mathbf{Hr^2}$			
f	fr^2	fr^3	r^3	$\mathbf{Hr^3}$
1	r^2	r	fr	**Hr**

Definition

The index of a subgroup H of G, written $[G : H]$, is the number of distinct left (or equivalently, right) cosets of H in G.

Lagrange's theorem

If H is a subgroup of finite group G, then $|G| = [G : H] \cdot |H|$. $\square$

The tower law

Proposition

Let G be a finite group and $K \leq H \leq G$ be a chain of subgroups. Then

$$[G : K] = [G : H][H : K].$$

Here is a “proof by picture”:

$[G : H] = \#$ of cosets of H in G

$[H : K] = \#$ of cosets of K in H

$[G : K] = \#$ of cosets of K in G

zH	z_1K	z_2K	z_3K	$\cdots$	z_nK
	$\vdots$	$\vdots$	$\vdots$	$\iddots$	$\vdots$
aH	a_1K	a_2K	a_3K	$\cdots$	a_nK
H	K	h_2K	h_3K	$\cdots$	h_nK

Proof

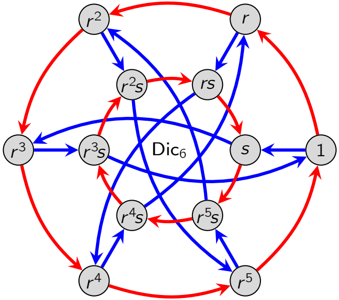
By Lagrange’s theorem,

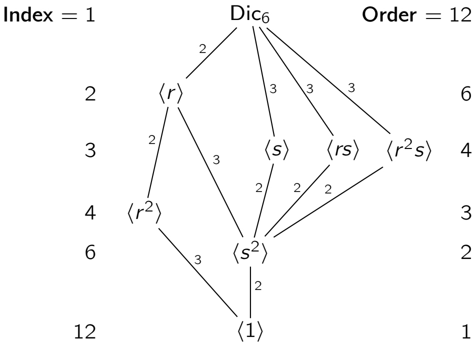
$$[G : H][H : K] = \frac{|G|}{|H|} \cdot \frac{|H|}{|K|} = \frac{|G|}{|K|} = [G : K]. \qquad \square$$

The tower law

Another way to visualize the tower law involves subgroup lattices.

It can be helpful to label the edge from H to K in a subgroup lattice with the index $[H : K]$.





The tower law and subgroup lattices

For any two subgroups $K \leq H$ of G, the index of K in H is just the *products of the edge labels* of any path from H to K.

Cosets in additive groups

In any abelian group, left cosets and right cosets coincide, because

$$xH = \{xh \mid h \in H\} = \{hx \mid h \in H\} = Hx.$$

In abelian groups written additively, like $\mathbb{Z}_n$ and $\mathbb{Z}$, left cosets are written not as aH, but

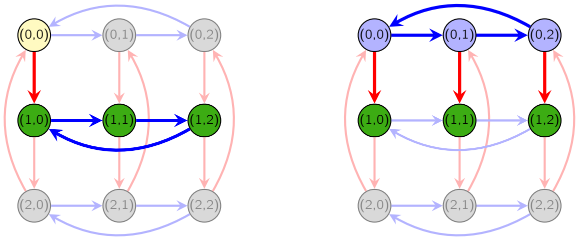
$$a + H = \{a + h \mid h \in H\}.$$

For example, let $G = \mathbb{Z}$. The cosets of the subgroup $H = 4\mathbb{Z} = \{4k \mid k \in \mathbb{Z}\}$ are

$$\begin{aligned} H &= \{\dots, -12, -8, -4, 0, 4, 8, 12, \dots\} = H \\ 1 + H &= \{\dots, -11, -7, -3, 1, 5, 9, 13, \dots\} = H + 1 \\ 2 + H &= \{\dots, -10, -6, -2, 2, 6, 10, 14, \dots\} = H + 2 \\ 3 + H &= \{\dots, -9, -5, -1, 3, 7, 11, 15, \dots\} = H + 3. \end{aligned}$$

Note that $3H$ would be interpreted to mean the subgroup $3(4\mathbb{Z}) = 12\mathbb{Z}$.

$(1,0) + \langle (0,1) \rangle$

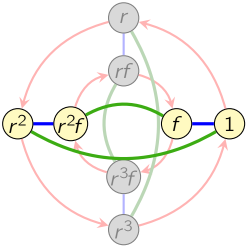


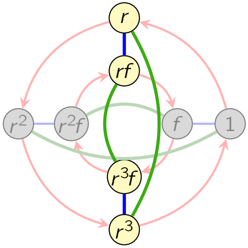
$\langle (0,1) \rangle + (1,0)$

Equality of sets vs. equality of elements

Caveat!

An equality of cosets $xH = Hx$ as sets *does not* imply an equality of elements $xh = hx$.





rH	r	r^3	rf	r^3f
H	1	r^2	f	r^2f

r	r^3	fr	fr^3	**Hr**
1	r^2	f	r^2f	**H**

Proposition

If $[G : H] = 2$, then both left cosets of H are also right cosets.

The center of a group

Even though $xH = Hx$ does not imply $xh = hx$ for all $h \in H$, the converse holds.

Even in a nonabelian group, there may be elements that commute with everything.

Definition

The center of G is the set

$$Z(G) = \{z \in G \mid gz = zg, \;\; \forall g \in G\}.$$

If $z \in Z(G)$, we say that z is central in G.

Examples

Let's think about what elements commute with everything in the following groups:

- $Z(D_4) = \langle r^2 \rangle = \{1, r^2\}$
- $Z(D_3) = \{1\}$
- $Z(Q_8) = \langle -1 \rangle = \{1, -1\}$
- $Z(\mathbf{Frz}_1) = \langle v \rangle = \{1, v\}$
- $Z(S_4) = \{e\}$
- $Z(A_4) = \{e\}$

Clearly, if $H \leq Z(G)$, then $xH = Hx$ for all $x \in G$.

The center of a group

Proposition

For any group G, the center $Z(G)$ is a subgroup.

Proof

- **Identity**: $eg = ge$ for all $g \in G$. ✓
- **Inverses**: Take $z \in Z(G)$. For any $g \in G$, we know that $zg = gz$.

  Multipy this on the left and right by z^{-1}:

  $$gz^{-1} = z^{-1}(zg)z^{-1} = z^{-1}(gz)z^{-1} = z^{-1}g.$$

  Therefore, $z^{-1} \in Z(G)$. ✓
- **Closure**: Suppose $z_1, z_2 \in Z(G)$. Then for any $g \in G$,

  $$(z_1z_2)g = z_1(z_2g) = z_1(gz_2) = (z_1g)z_2 = (gz_1)z_2 = g(z_1z_2).$$

  Therefore, $z_1z_2 \in Z(G)$. ✓