Visual Algebra

Lecture 3.4: Normalizers and normal subgroups

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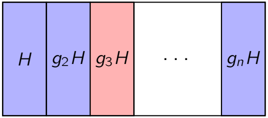
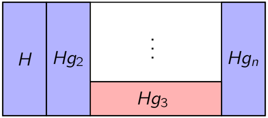
Paritions by left vs. right cosets

Given a subgroup H of G, it is natural to ask the following question:

"*How many left cosets of H are right cosets?*"



Partition of G by the left cosets of H



Partition of G by the right cosets of H

- "Best case" scenario: all of them
- "Worst case" scenario: only H
- In general: somewhere between these two extremes

Definition

A subgroup H is a normal subgroup of G if $gH = Hg$ for *all* $g \in G$. We write $H \trianglelefteq G$.

The normalizer of H, denoted $N_G(H)$, is the set of elements $g \in G$ such that $gH = Hg$:

$$N_G(H) = \{g \in G \mid gH = Hg\},$$

i.e., the union of left cosets that are also right cosets.

Examples of normal sugroups

We've seen cases where we know a subgroup will be normal without having to check.

1. The subgroup $H = G$ is always normal. The only left coset is also the only right coset:
$$eG = G = Ge.$$
2. The subgroup $H = \{e\}$ is always normal. The left and right cosets are singletons sets:
$$gH = \{g\} = Hg.$$
3. Subgroups H of index 2 are normal. The two cosets (left or right) are H and $G - H$.
4. Subgroups of *abelian groups* are always normal, because for any $H \leq G$,
$$aH = \big\{ah \mid h \in H\big\} = \big\{ha \mid h \in H\big\} = Ha.$$
5. Subgroups $H \leq Z(G)$ are always normal, for the same reason as above.

Normalizers are subgroups

Theorem

For any $H \leq G$, we have $N_G(H) \leq G$.

Proof

- **Identity**: $eH = He$. ✓
- **Inverses**: Suppose $gH = Hg$. Multiply on the left and right by g^{-1}:

$$Hg^{-1} = g^{-1}(gH)g^{-1} = g^{-1}(Hg)g^{-1} = g^{-1}H. \quad ✓$$

- **Closure**: Suppose $g_1H = Hg_1$ and $g_2H = Hg_2$. Then

$$(g_1g_2)H = g_1(g_2H) = g_1(Hg_2) = (g_1H)g_2 = (Hg_1)g_2 = H(g_1g_2). \quad ✓$$

Corollary

Every subgroup is normal in its normalizer:

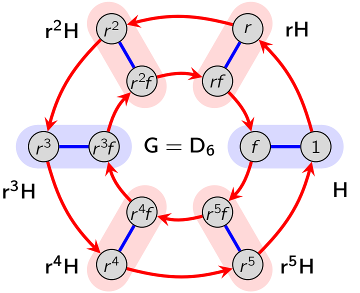
$$H \trianglelefteq N_G(H) \leq G.$$

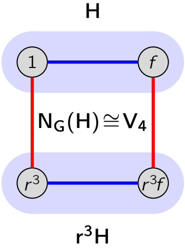
Proof

By definition, $gH = Hg$ for all $g \in N_G(H)$. Therefore, $H \trianglelefteq N_G(H)$. □

How to spot the normalizer in a Cayley graph

If we "collapse" G by the left cosets of H and disallow H-arrows, then $N_G(H)$ consists of the cosets that are reachable from H by a unique path.





We can get from H to rH multiple ways: via r or r^5.

The *only* way to get from H to r^3H is via the path r^3.

Remark

The normalizer of the subgroup $H = \langle f \rangle$ of D_n is

$$N_{D_n}(H) = \begin{cases} H \cup r^{n/2}H = \{1, f, r^{n/2}, r^{n/2}f\} & n \text{ even} \\ H = \{1, f\} & n \text{ odd.} \end{cases}$$

Conjugate subgroups

For a fixed element $g \in G$, the set

$$gHg^{-1} = \{ghg^{-1} \mid h \in H\}$$

is called the conjugate of H by g.

Observation 1

For any $g \in G$, the conjugate gHg^{-1} is a subgroup of G.

Proof

1. **Identity**: $e = geg^{-1}$. ✓
2. **Closure**: $(gh_1g^{-1})(gh_2g^{-1}) = gh_1h_2g^{-1}$. ✓
3. **Inverses**: $(ghg^{-1})^{-1} = gh^{-1}g^{-1}$. ✓

Observation 2

$gh_1g^{-1} = gh_2g^{-1}$ if and only if $h_1 = h_2$. □

Later, we'll prove that H and gHg^{-1} are isomorphic subgroups.

How to check if a subgroup is normal

If $gH = Hg$, then right-multiplying both sides by g^{-1} yields $gHg^{-1} = H$.

This gives us a new way to check whether a subgroup H is normal in G.

Useful remark

The following are equivalent to a subgroup $H \leq G$ being normal:

(i) $gH = Hg$ for all $g \in G$; ("*left cosets are right cosets*")

(ii) $gHg^{-1} = H$ for all $g \in G$; ("*only one conjugate subgroup*")

(iii) $ghg^{-1} \in H$ for all $h \in H$, $g \in G$. ("*closed under conjugation*")

Proof

(i)$\Leftrightarrow$(ii)$\Rightarrow$(iii) is trivial.

(iii)$\Rightarrow$(ii) is an exercise. □

Sometimes, one of these is *much* easier than the others! For example:

- to show $H \ntrianglelefteq G$, find *one element* $h \in H$ for which $ghg^{-1} \notin H$ for some $g \in G$.
- if G has a unique subgroup of size $|H|$, then H *must* be normal. (*Why?*)

A curious example

Useful remark

The following are equivalent to a subgroup $H \leq G$ being normal:

(i) $gH = Hg$ for all $g \in G$; ("*left cosets are right cosets*")

(ii) $gHg^{-1} = H$ for all $g \in G$; ("*only one conjugate subgroup*")

(iii) $ghg^{-1} \in H$ for all $h \in H$, $g \in G$. ("*closed under conjugation*")

If G is infinite, then $gHg^{-1} \subsetneq H$ is possible, but only if $H \ntrianglelefteq G$. (*Why?*)

In geometric group theory, the Baumslag-Solitar groups are defined by

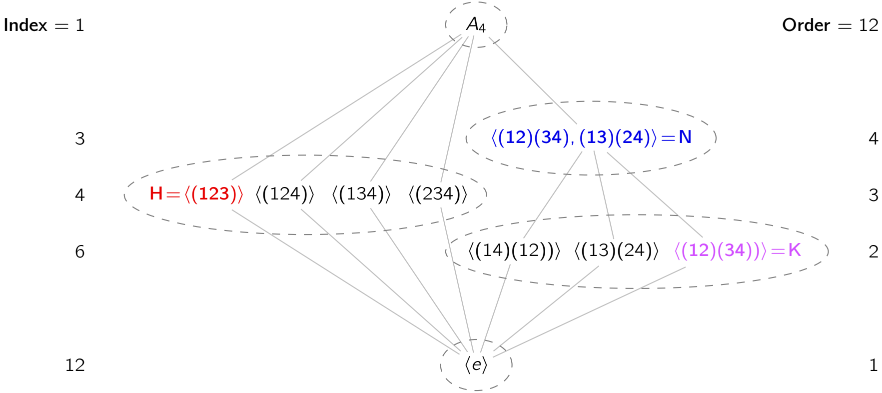
$$\mathrm{BS}(m,n) = \langle a, b \mid ba^m b^{-1} = a^n \rangle \cong \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{n}{m} & 0 \\ 0 & 1 \end{bmatrix} \right\rangle .$$

Consider the group $G = \mathrm{BS}(1,2)$:

$$G = \langle a, b \mid bab^{-1} = a^2 \rangle, \qquad H = \langle a \rangle .$$

Note that $bHb^{-1} = b\langle a \rangle b^{-1} = \langle bab^{-1} \rangle = \langle a^2 \rangle \lneq \langle a \rangle = H$, so $H \ntrianglelefteq G$.

The subgroup lattice of A_4



Going forward, we will consider the following three subgroups of A_4:

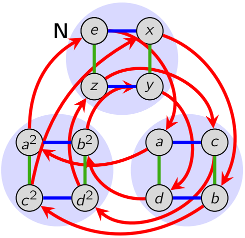
$$N = \langle (12)(34), (13)(24) \rangle = \{e,\ (12)(34),\ (13)(24),\ (14)(23)\} \cong V_4$$
$$H = \langle (123) \rangle = \{e,\ (123),\ (132)\} \cong C_3$$
$$K = \langle (12)(34) \rangle = \{e,\ (12)(34)\} \cong C_2.$$

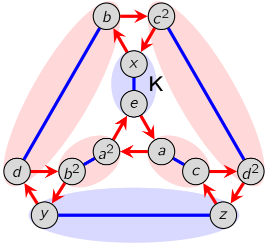
For each one, its normalizer lies between it and A_4 (inclusive) on the subgroup lattice.

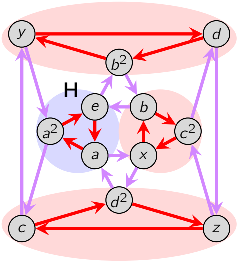
Three subgroups of A_4

The normalizer of each subgroup consists of the elements in the blue left cosets.

Here, take $a = (123)$, $x = (12)(34)$, $z = (13)(24)$, and $b = (234)$.







(124)	(234)	(143)	(132)
(123)	(243)	(142)	(134)
e	(12)(34)	(13)(24)	(14)(23)

$[A_4 : N_{A_4}(N)] = 1$

"normal"

(124)	(234)	(143)	(132)
(123)	(243)	(142)	(134)
e	(12)(34)	(13)(24)	(14)(23)

$[A_4 : N_{A_4}(K)] = 3$

"moderately unnormal"

(14)(23)	(142)	(143)
(13)(24)	(243)	(124)
(12)(34)	(134)	(234)
e	(123)	(132)

$[A_4 : N_{A_4}(H)] = 4$

"fully unnormal"

The degree of normality

Let $H \leq G$ have index $[G : H] = n < \infty$. Let's define a term that describes:

"*the proportion of cosets that are blue*"

Definition

Let $H \leq G$ with $[G : H] = n < \infty$. The **degree of normality** of H is

$$\text{Deg}_G^{\triangleleft}(H) := \frac{|N_G(H)|}{|G|} = \frac{1}{[G : N_G(H)]} = \frac{\#\text{ elements } x \in G \text{ for which } xH = Hx}{\#\text{ elements } x \in G}.$$

- If $\text{Deg}_G^{\triangleleft}(H) = 1$, then H is normal.
- If $\text{Deg}_G^{\triangleleft}(H) = \frac{1}{n}$, we'll say H is fully unnormal.
- If $\frac{1}{n} < \text{Deg}_G^{\triangleleft}(H) < 1$, we'll say H is moderately unnormal.

Big idea

The degree of normality measures *how close to being normal* a subgroup is.