Visual Algebra

Lecture 3.6: Conjugate subgroups

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Normal subgroups of order 2

Often, we can determine the normal subgroups and conjugacy classes simply from inspecting the subgroup lattice.

We'll make frequent use of the following straightforward result.

Lemma

A subgroup H of order 2 is normal if and only if it is contained in $Z(G)$.

Proof

Let $H = \{e, h\}$.

"$\Leftarrow$": We already know that subgroups contained in $Z(G)$ are normal. ✓

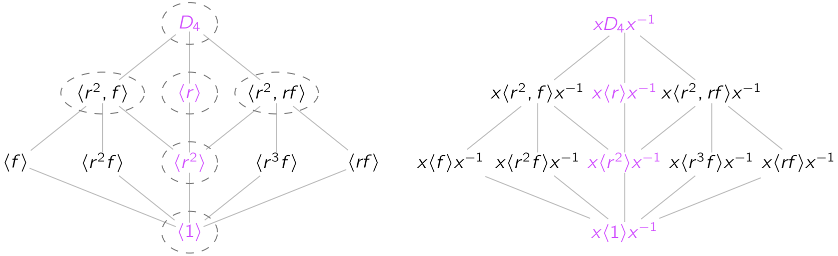
"$\Rightarrow$": Suppose $H \trianglelefteq G$. Then for all $x \in G$,

$$xH = x\{e, h\} = \{x, xh\}, \quad \text{and} \quad Hx = \{e, h\}x = \{x, hx\}.$$

Since $xH = Hx$, we must have $xh = hx$, and hence $H \leq Z(G)$. ✓

Unicorn subgroups

Suppose we conjugate $G = D_4$ by some element $x \in D_4$.



Subgroups at a unique "lattice neighborhood" are called unicorns, and must be normal.

For example, $\langle r^2\rangle = x\langle r^2\rangle x^{-1}$ is the only size-2 subgroup "*with* 3 *parents*."

The groups G and $\langle 1\rangle$ are always unicorns, and hence normal.

The index-2 subgroups $\langle r^2, f\rangle$, $\langle r\rangle$, and $\langle r^2, rf\rangle$ must be normal.

Remark

Conjugating a normal subgroup $N \leq G$ by $x \in G$ shuffles its elements and subgroups.

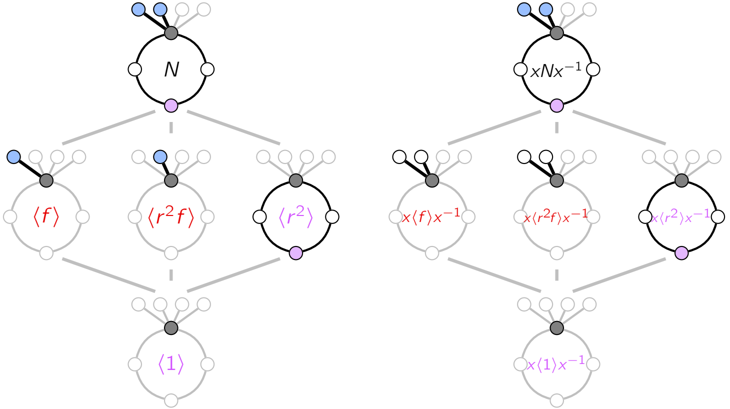
Conjugating normal subgroups

Proposition

If $H \leq N \trianglelefteq G$, then $xHx^{-1} \leq N$ for all $x \in G$.

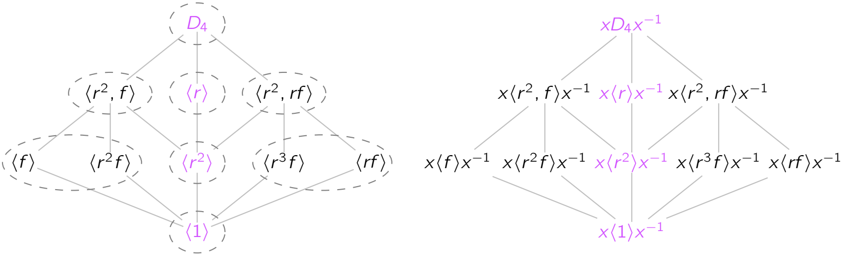
Proof

Conjugating $H \leq N$ by $x \in G$ yields $xHx^{-1} \leq xNx^{-1} = N$. □



Determining the conjugacy classes from the subgroup lattice

Suppose we conjugate $G = D_4$ by some element $x \in D_4$.



Conclusions

- All unicorns and index-2 subgroups are normal.
- $\langle f \rangle$ cannot be normal because $f \notin Z(D_4)$. Thus, it has some other conjugate.
- Each conjugate to $\langle f \rangle$ must be contained in $\langle r^2, f \rangle$. Therefore,

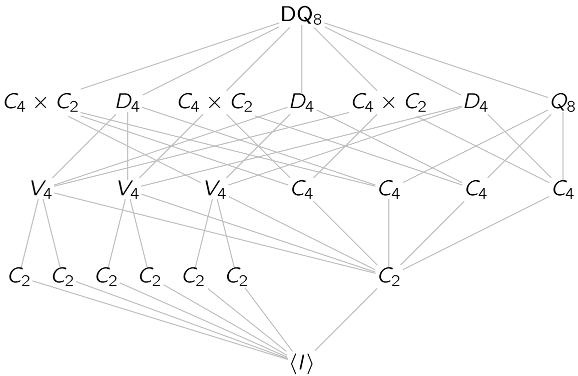
$$\mathrm{cl}_{D_4}(\langle f \rangle) = \big\{\langle f \rangle, \langle rf \rangle\big\} = \mathrm{cl}_{D_4}(\langle rf \rangle).$$

- The normalizer of $\langle f \rangle$ must have index 2, and thus $N_{D_4}(\langle f \rangle) = \langle r^2, f \rangle$.
- *We just determined all conjugacy classes and normalizers simply by inspection!*

Unicorns in the diquaternion group

Our definition of unicorn could be strengthened, but we want to keep things simple.

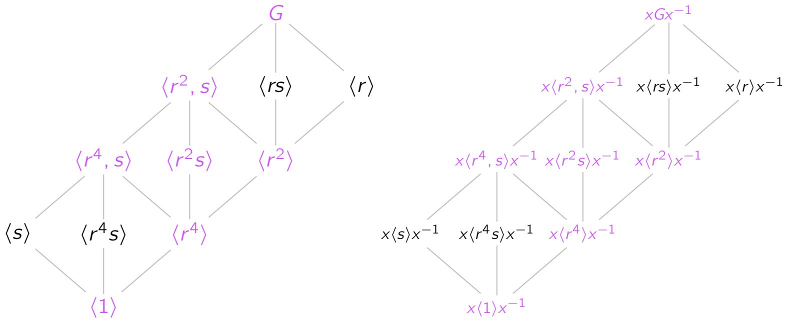
Are any of the C_4 subgroups of DQ_8 unicorns, i.e., "not like the others"?



What can we say about the conjugacy classes of the subgroups of DQ_8 just from the lattice?

A mystery group of order 16

Let's repeat a previous exercise, for this lattice of an actual group. Unicorns are purple.



We can deduce that every subgroup is normal, except possibly $\langle s\rangle$ and $\langle r^4 s\rangle$.

There are two cases:

- $\langle s\rangle$ and $\langle r^4 s\rangle$ are normal $\Rightarrow s \in Z(G) \Rightarrow G$ is abelian.
- $\langle s\rangle$ and $\langle r^4 s\rangle$ are not normal $\Rightarrow \mathrm{cl}_G(\langle s\rangle) = \big\{\langle s\rangle, \langle r^4 s\rangle\big\} \Rightarrow G$ is nonabelian.

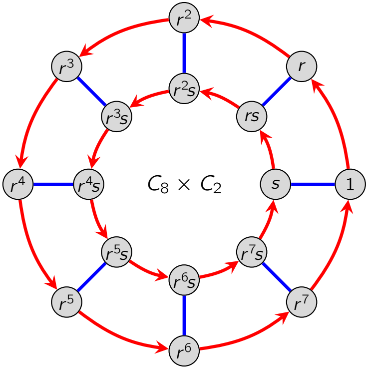
This doesn't necessarily mean that both of these are actually possible...

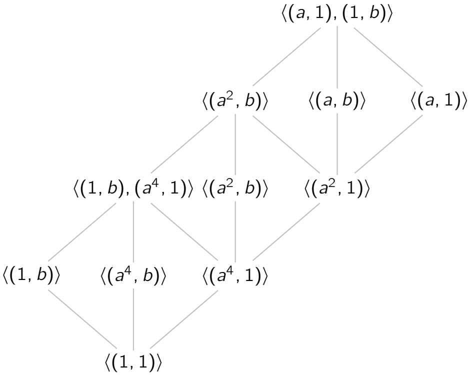
A mystery group of order 16

It's straightforward to check that this is the subgroup lattice of

$$C_8 \times C_2 = \langle r, s \mid r^8 = s^2 = 1, srs = r \rangle.$$

Let $r = (a, 1)$ and $s = (1, b)$, and so $C_8 \times C_2 = \langle r, s \rangle = \langle (a, 1), (1, b) \rangle$.

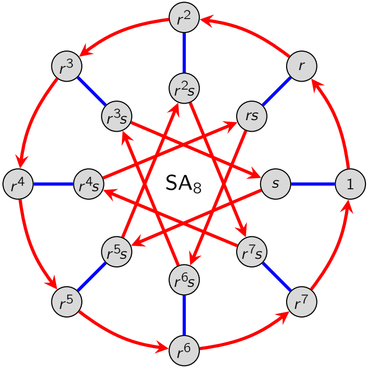


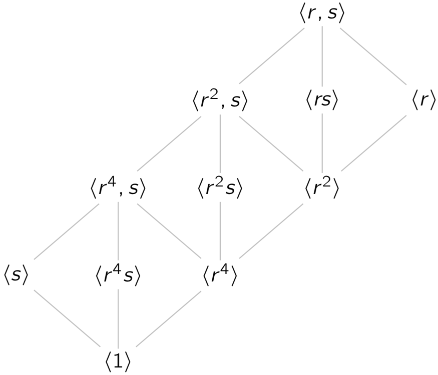


A mystery group of order 16

However, the nonabelian case is possible as well! The following also works:

$$\mathrm{SA}_8 = \langle r, s \mid r^8 = s^2 = 1, srs = r^5 \rangle.$$





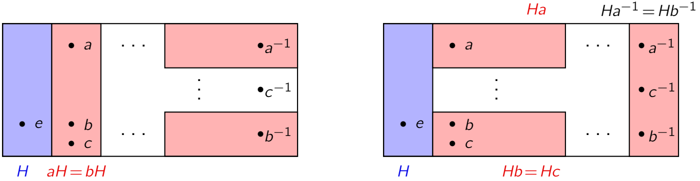
Conjugate subgroups, algebraically

We understand how to compare gH and Hg both algebraically and in a Cayley graph.

But to understand H vs. gHg^{-1}, we need to compare gH to Hg^{-1}.

Proposition

If $aH = bH$, then $Ha^{-1} = Hb^{-1}$.



Proof

Using $x \in H \Leftrightarrow xH = H = Hx$, we deduce that

$$aH = bH \quad \Leftrightarrow \quad b^{-1}aH = H \quad \Leftrightarrow \quad H = Hb^{-1}a \quad \Leftrightarrow \quad Ha^{-1} = Hb^{-1}.$$

(Note that we're taking $x = b^{-1}a$ above.) □

Conjugate subgroups, algebraically

We just showed that $aH = bH$ implies $Ha^{-1} = Hb^{-1}$.

Corollary

If $aH = bH$, then $aHa^{-1} = bHb^{-1}$.

Proof

Since $aH = bH$ we know that $Ha^{-1} = Hb^{-1}$, and so

$$aHa^{-1} = (aH)a^{-1} = (bH)a^{-1} = b(Ha^{-1}) = bHb^{-1}. \qquad \square$$

Corollary

For any subgroup $H \leq G$ of finite index, there are at most $[G : H]$ conjugates of H. $\square$

In summary, we have

$$\big|\text{cl}_G(H)\big| = [G : N_G(H)] \leq [G : H].$$

We proved the inequality, but the equality remains unproven. (We'll wait for group actions.)

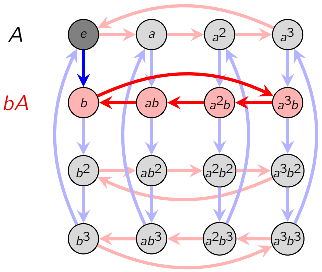
Conjugate subgroups, visually

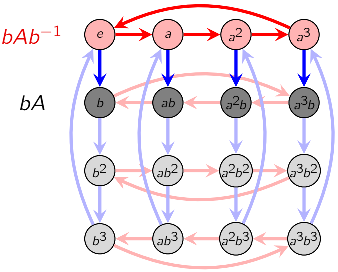
Remark

To identify the conjugate subgroup gHg^{-1} in the Cayley graph, do the following:

1. Identify the left coset gH,
2. From each node in gH, traverse the g^{-1}-path.

Here is an example of this for the normal subgroup $A = \langle a \rangle$ of $G = C_4 \rtimes C_4$.





Let's check that $b^2Ab^{-2} = A$ and $b^3Ab^{-3} = A$, which means that $A \trianglelefteq G$.

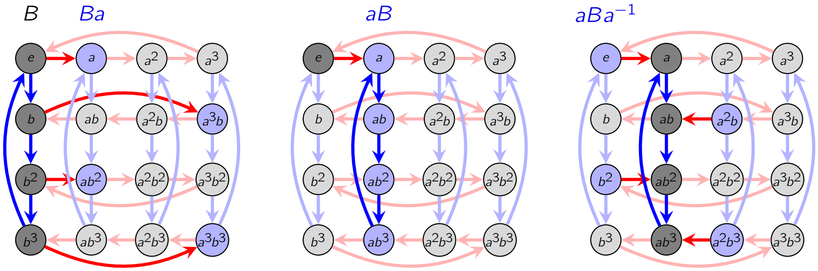
Conjugate subgroups, visually

Remark

To identify the conjugate subgroup gHg^{-1} in the Cayley graph, do the following:

1. Identify the left coset gH,
2. From each node in gH, traverse the g^{-1}-path.

Let's carry out the same steps with the nonnormal subgroup $A = \langle B \rangle$ of $G = C_4 \rtimes C_4$.



It follows immediately that B is not normal. Let's find all conjuguate subgroups...

Conjugate subgroups, visually

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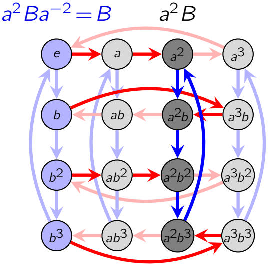
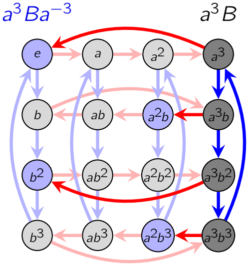
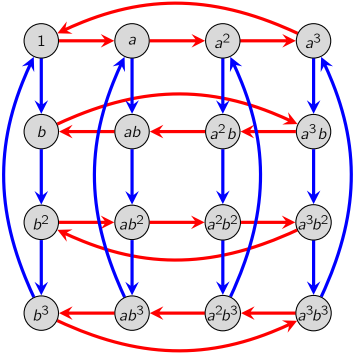


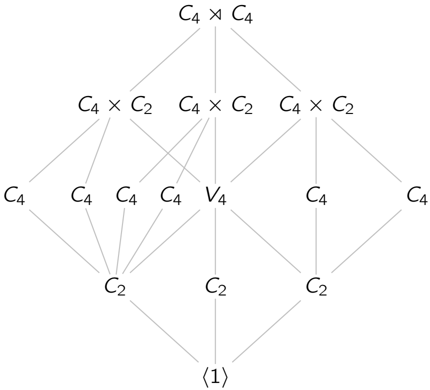


We conclude that $\text{cl}_G(B) = \{B, aBa^{-1}\}$.

It follows that $[G : N_G(B)] = 2$, i.e., $|N_G(B)| = 8$. By inspection, $N_G(B) = B \cup a^2B$.

The subgroup lattice of $C_4 \rtimes C_4$





Exercises

- Draw the subgroup lattice with the subgroups defined by generators.
- Determine the conjugacy classes of subgroups.
- Construct a cycle graph.