Visual Algebra

Lecture 3.7: Products of subgroups

Dr. Matthew Macauley

School of Mathematical & Statistical Sciences
Clemson University
South Carolina, USA
http://www.math.clemson.edu/~macaule/

The product of two subgroups

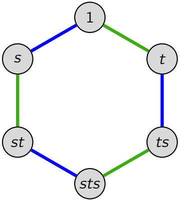
We have seen a number of definitions that involve a product of elements and subgroups:

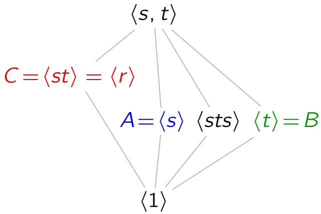
- Left cosets: $xH = \{xh \mid h \in H\}$
- Right cosets: $Hx = \{hx \mid h \in H\}$
- Conjugate subgroups: $xHx^{-1} = \{xhx^{-1} \mid h \in H\}$.

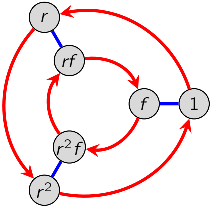
We can also define the product of two subgroups $A, B \leq G$:

$$AB = \{ab \mid a \in A,\ b \in B\}.$$

Let's investigate when this is a subgroup.







Notice that

$$AB = \{1, s, t, st\} \not\leq D_3, \qquad AC = \{1, r, r^2, f, fr, fr^2\} = D_3.$$

When is AB a subgroup?

Observation

If $AB = \{ab \mid a \in A,\ b \in B\}$ is a subgroup, then it must be "above" A and B in the lattice.

For closure to hold in AB, we need $(a_1 b_1)(a_2 b_2) \in AB$. It suffices to have $b_1 a_2 \in AB$.

Remark

If $A \leq N_G(B)$, "*A normalizes B*", i.e.,

$$\{ab \mid b \in B\} = aB = Ba = \{b'a \mid b' \in B\},$$

then every $ab \in AB$ can be written as some $b'a \in BA$.

Suppose A normalizes B. Then

$$(a_1 b_1)(a_2 b_2) = a_1(b_1 a_2)b_2 = a_1(a_2 b_1')b_2 \in AB.$$

Proposition

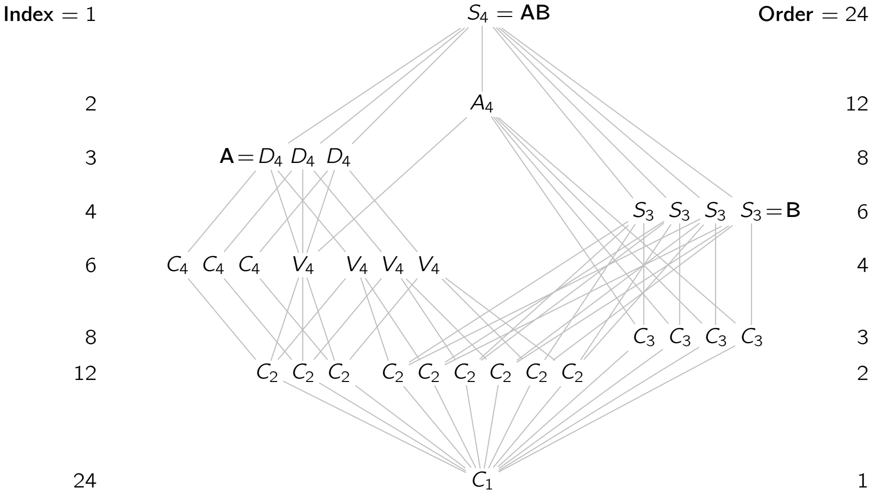
If $A, B \leq G$ and one normalizes the other, then AB is a subgroup of G.

In particular, *if at least one of them is normal, then $AB \leq G$*. □

When is AB a subgroup?

It may still happen that $AB = G$, even if neither subgroup normalizes the other.

For example, in $G = S_4$, subgroups $A \cong D_4$ and $B \cong S_3$ are their own normalizers.



Double cosets

Definition

If $A, B \leq G$ and $x \in G$, an (A, B)-double coset is a set

$$AxB := \{axb \mid a \in A,\ b \in B\}.$$

Proposition (Chapter 5 exercise)

(i) Even if AB $(= AeB)$ is not a subgroup, it has size

$$|AB| = \frac{|A| \cdot |B|}{|A \cap B|} = [A : A \cap B] \cdot |B|.$$

(ii) $x \sim y$ iff $x \in AyB$ is an equivalence relation.

(iii) G is the disjoint union of its (A, B)-double cosets.

(iv) AxB is the union of exactly $[A : A \cap xBx^{-1}]$ left cosets of B in G.

(v) The size of the double coset AxB is

$$|AxB| = (\#\text{ left cosets of } B \text{ in } AxB) \cdot |B| = [A : A \cap xBx^{-1}] \cdot |B|.$$

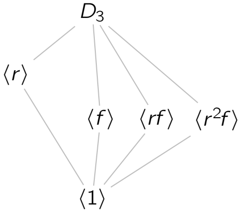
(vi) The size of the double coset AxB is

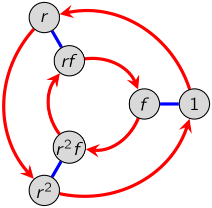
$$|AxB| = [A : A \cap xBx^{-1}] \cdot |B| = \frac{|A| \cdot |B|}{|A \cap xBx^{-1}|}.$$

Double cosets in D_3

Let's compute the double cosets of $A = \{1, rf\}$ and $B = \{1, f\}$ in D_3.

$$AB = \{1, f, rf, r\}, \qquad Ar^2B = \{r^2, r^2f\}.$$





f	r	r^2
1	rf	r^2f

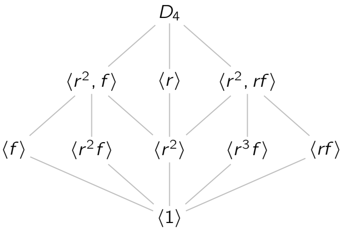
Recall that

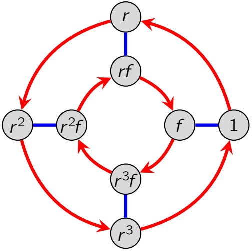
$$|AB| = [A : A \cap B] \cdot |B| = \frac{|A| \cdot |B|}{|A \cap B|}, \qquad |AxB| = [A : A \cap xBx^{-1}] \cdot |B| = \frac{|A| \cdot |B|}{|A \cap xBx^{-1}|},$$

Double cosets in D_4

Let's compute the double cosets of $A = \{1, r^2 f\}$ and $B = \{1, f\}$ in D_4.

$$AB = \{1, f, r^2, r^2 f\}, \qquad ArB = \{r, rf\}, \qquad Ar^3B = \{r^3, r^3 f\}.$$





$r^3 f$	r^3
r	rf
r^2	$r^2 f$
1	f

Recall that

$$|AB| = [A : A \cap B] \cdot |B| = \frac{|A| \cdot |B|}{|A \cap B|}, \qquad |AxB| = [A : A \cap xBx^{-1}] \cdot |B| = \frac{|A| \cdot |B|}{|A \cap xBx^{-1}|},$$

Remark

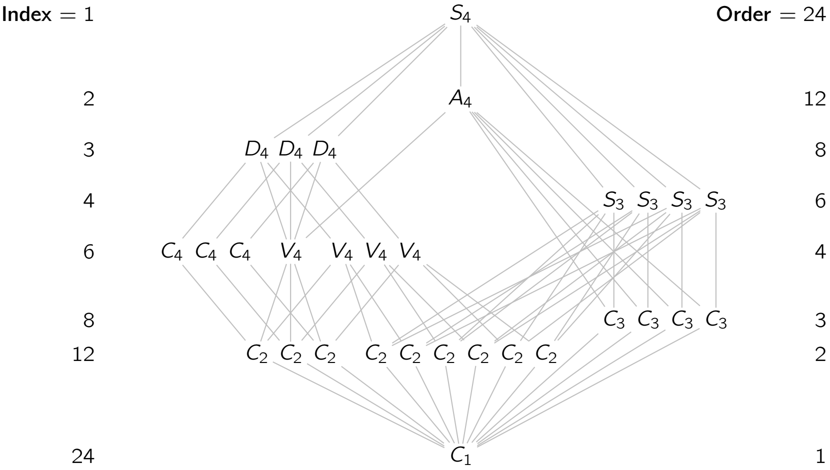
If $|A| \cdot |B| > |G|$, then $A \cap B \neq \{e\}$.

Revisiting S_4

With the knowledge that

$$|AB| = [A : A \cap B] \cdot |B| = \frac{|A| \cdot |B|}{|A \cap B|}, \qquad |AxB| = [A : A \cap xBx^{-1}] \cdot |B| = \frac{|A| \cdot |B|}{|A \cap xBx^{-1}|},$$

think about what AB is for various subgroups.



Revisiting A_5