Visual Algebra

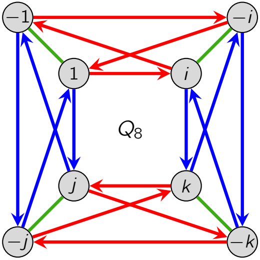
Lecture 3.8: Quotient groups

Dr. Matthew Macauley

School of Mathematical & Statistical Sciences
Clemson University
South Carolina, USA
http://www.math.clemson.edu/~macaule/

Quotients and cosets

We have already encountered the concept a quotient of a group by a subgroup:



	1	-1	i	$-i$	j	$-j$	k	$-k$
1	1	-1	i	$-i$	j	$-j$	k	$-k$
-1	-1	1	$-i$	i	$-j$	j	$-k$	k
i	i	$-i$	-1	1	k	$-k$	$-j$	j
$-i$	$-i$	i	1	-1	$-k$	k	j	$-j$
j	j	$-j$	$-k$	k	-1	1	i	$-i$
$-j$	$-j$	j	k	$-k$	1	-1	$-i$	i
k	k	$-k$	j	$-j$	$-i$	i	-1	1
$-k$	$-k$	k	$-j$	j	i	$-i$	1	-1

$Q_8/\langle -1\rangle \cong V_4$

	± 1	$\pm i$	$\pm j$	$\pm k$
± 1	± 1	$\pm i$	$\pm j$	$\pm k$
$\pm i$	$\pm i$	± 1	$\pm k$	$\pm j$
$\pm j$	$\pm j$	$\pm k$	± 1	$\pm i$
$\pm k$	$\pm k$	$\pm j$	$\pm i$	± 1

We now know enough algebra to be able to formalize this.

Key idea

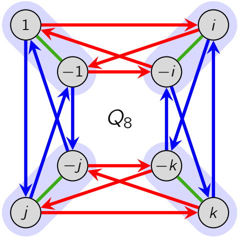
The quotient of G by a subgroup H exists when the (left) cosets of H form a group.

The "quotient process"

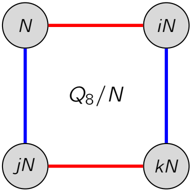
Goals

- Characterize *when* a quotient exists.
- Learn *how* to formalize this algebraically (without Cayley graphs or tables).

First, let's interpret the "*quotient process*" visually, in terms of cosets.



Cluster the left cosets of N



Collapse cosets into single nodes

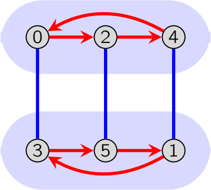
	N	iN	jN	kN
N	N	iN	jN	kN
iN	iN	N	kN	jN
jN	jN	kN	N	iN
kN	kN	jN	iN	N

Elements of the quotient are cosets of N

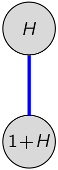
Notice how taking a quotient generally loses information.

Can you think of two $G_1 \not\cong G_2$ for which $G_1/N \cong G_2/N$?

Two examples of a quotient



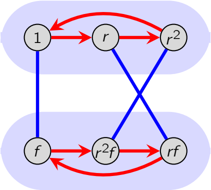
Cluster the
left cosets of $H \leq \mathbb{Z}_6$



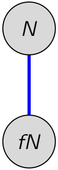
Collapse cosets
into single nodes

	H	$1+H$
H	H	$1+H$
$1+H$	$1+H$	H

Elements of the quotient
are cosets of H



Cluster the
left cosets of $N \leq D_3$



Collapse cosets
into single nodes

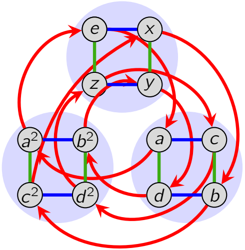
	N	fN
N	N	fN
fN	fN	N

Elements of the quotient
are cosets of N

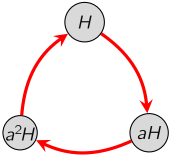
We say that $\mathbb{Z}_6/\langle 2\rangle \cong \mathbb{Z}_2$ and $D_3/\langle r\rangle \cong C_2$.

Another example of a auotient

The quotient process succeeds for the group $N = \langle (12)(34), (13)(24) \rangle$ of A_4.



Cluster the left cosets of $H \leq A_4$



Collapse cosets into single nodes

	H	aH	a^2H
H	H	aH	a^2H
aH	aH	a^2H	H
a^2H	a^2H	H	aH

Elements of the quotient are cosets of H

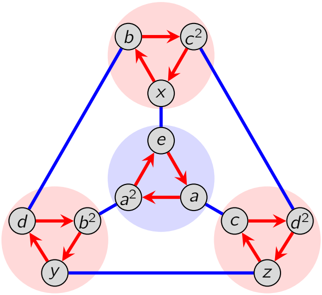
We denote the resulting group by $G/N = \{N,\ aN,\ a^2N\} \cong C_3$. Since it's a group, there is a binary operation on the set of cosets of N.

Questions

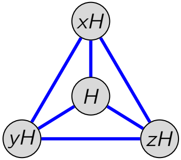
- Do you see *how* to define this binary operation?
- Do you see *why* this works for this particular $N \leq G$?
- Can you think of examples where this "quotient process" would fail, and why?

A non-example of a quotient

The quotient process fails for the group $H = \langle (123) \rangle$ of A_4.



Cluster the left cosets of $H = \langle (123) \rangle$.



Collapse cosets into single nodes

We can still write $G/H := \{H,\ xN,\ yH,\ zH\}$ for the set of (left) cosets of H in G.

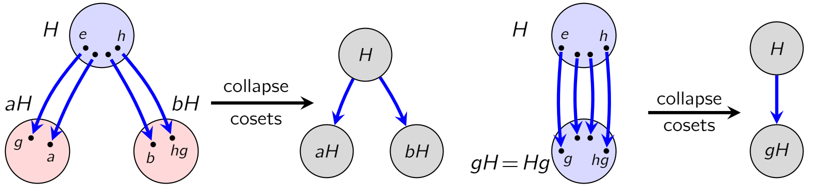
However, the resulting graph is not the Cayley graph of a group.

In other words, something goes wrong if we try to define a binary operaton on G/H.

When and why the quotient process works

To get some intuition, let's consider collapsing the left cosets of a subgroup $H \leq G$.

In the following: *the right coset Hg are the "arrowtips"*.



Elements in the right coset Hg are in multiple left cosets

not a valid Cayley graph

Elements in Hg all stay in gH

valid Cayley graph

Key idea

If H is normal subgroup of G, then the quotient group G/H exists.

If H is not normal, then following the blue arrows from H is ambiguous.

In other words, it depends on our where we start within H.

We still need to formalize this and prove it algebraically.

What does it mean to "multiply" two cosets?

Quotient theorem

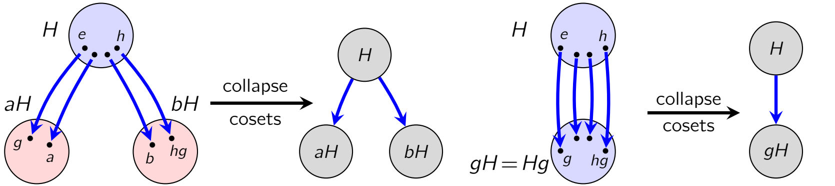
If $H \trianglelefteq G$, the set of cosets G/H forms a group, with binary operation

$$aH \cdot bH := abH.$$

It is clear that G/H is closed under this operation.

We have to show that this operation is well-defined.

By that, we mean that it *does not depend on our choice of coset representative*.



our destination depends on where in H we start

the quotient process does not yield a group

destination doesn't depend on where we start in H

quotient process succeeds

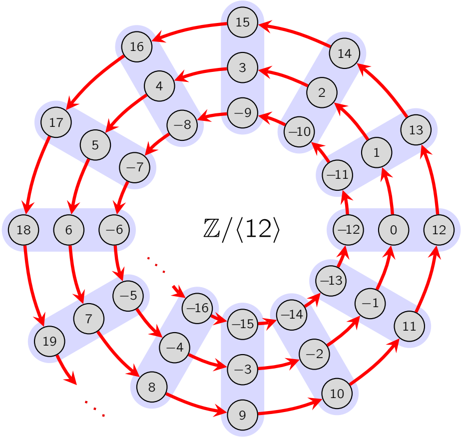
A familiar example

Consider the subgroup $H = \langle 12 \rangle = 12\mathbb{Z}$ of $G = \mathbb{Z}$.

The cosets of H are the congruence classes modulo 12.

Since this group is additive, the condition $aH \cdot bH$ becomes $(a + H) + (b + H) = a + b + H$:

"(the coset containing a) + (the coset containing b) = the coset containing $a + b$."



Quotient groups, algebraically

Lemma

Let $H \trianglelefteq G$. Multiplication of cosets is well-defined:

$$\text{if } a_1H = a_2H \text{ and } b_1H = b_2H, \text{ then } a_1H \cdot b_1H = a_2H \cdot b_2H.$$

Proof

Suppose that $H \trianglelefteq G$, $a_1H = a_2H$ and $b_1H = b_2H$. Then

$$\begin{aligned}
a_1H \cdot b_1H &= a_1b_1H && \text{(by definition)} \\
&= a_1(b_2H) && (b_1H = b_2H \text{ by assumption}) \\
&= (a_1H)b_2 && (b_2H = Hb_2 \text{ since } H \trianglelefteq G) \\
&= (a_2H)b_2 && (a_1H = a_2H \text{ by assumption}) \\
&= a_2b_2H && (b_2H = Hb_2 \text{ since } H \trianglelefteq G) \\
&= a_2H \cdot b_2H && \text{(by definition)}
\end{aligned}$$

Thus, the binary operation on G/H is well-defined. □

Quotient groups, algebraically

Quotient theorem (restated)

When $H \trianglelefteq G$, the set of cosets G/H forms a group.

Proof

There is a well-defined binary operation on the set of left (equivalently, right) cosets:

$$aH \cdot bH = abH.$$

We need to verify the three remaining properties of a group:

Identity. The coset $H = eH$ is the identity because for any coset $aH \in G/H$,

$$aH \cdot H = aH \cdot eH = aeH = aH = eaH = eH \cdot aH = H \cdot aH. \quad \checkmark$$

Inverses. Given a coset aH, its inverse is $a^{-1}H$, because

$$aH \cdot a^{-1}H = aa^{-1}H = eH = a^{-1}aH = a^{-1}H \cdot aH. \quad \checkmark$$

Closure. This is immediate, because $aH \cdot bH = abH$ is another coset in G/H. $\checkmark$

Quotient groups, algebraically

We just learned that if $H \trianglelefteq G$, then we can define a binary operation on cosets by

$$aH \cdot bH = abH,$$

and *this works*.

Here's another reason why this makes sense.

Given any subgroup $H \leq G$, normal or not, define the product of left cosets:

$$xHyH = \big\{xh_1yh_2 \mid h_1, h_2 \in H\big\}.$$

Exercise

If H is normal, then the set $xHyH$ is equal to the left cosets

$$xyH = \big\{xyh \mid h \in H\big\}.$$

To show that $xHyH = xyH$, it suffices to verify that $\subseteq$ and $\supseteq$ both hold. That is:

- every element of the form xh_1yh_2 can be written as xyh for some $h \in H$.
- every element of the form xyh can be written as xh_1yh_2 for some $h_1, h_2 \in H$.

Note that one containment is trivial. This will be left for homework.

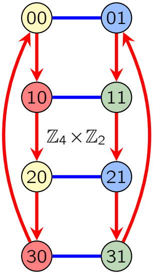
One last word on quotients

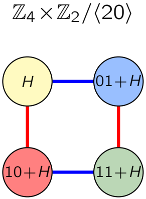
Remark

Do you think the following should be true or false, for subgroups H and K?

1. Does $H \cong K$ imply $G/H \cong G/K$?
2. Does $G/H \cong G/K$ imply $H \cong K$?
3. Does $H \cong K$ and $G_1/H \cong G_2/K$ imply $G_1 \cong G_2$?

All are false. Counterexamples for all of these can be found using the group $G = \mathbb{Z}_4 \times \mathbb{Z}_2$:





	H	$10+H$	$01+H$	$11+H$
H	H	$10+H$	$01+H$	$11+H$
$10+H$	$10+H$	H	$11+H$	$01+H$
$01+H$	$01+H$	$11+H$	H	$10+H$
$11+H$	$11+H$	$01+H$	$10+H$	H