Visual Algebra

Lecture 3.9: Conjugate elements

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Conjugate elements

We've seen how conjugation defines an equivalence relation on the set of subgroups of G.

The equivalence class containing $H \leq G$ is its conjugacy class, denoted $\mathrm{cl}_G(H)$.

We can also conjugate elements. Given $h \in G$, we may ask:

"*which elements can be written as xhx^{-1} for some $x \in G$?*"

Definition

The conjugacy class of an element $h \in G$ is the set

$$\mathrm{cl}_G(h) = \{xhx^{-1} \mid x \in G\}.$$

Proposition

The conjugacy class of $h \in G$ has size 1 if and only if $h \in Z(G)$.

Proof

Suppose $|\,\mathrm{cl}_G(h)| = 1$. This means that

$$\mathrm{cl}_G(h) = \{h\} \iff xhx^{-1} = h, \ \forall x \in G \iff xh = hx, \ \forall x \in G \iff h \in Z(G). \qquad \square$$

Conjugate elements

Lemma (exercise)

Conjugacy of elements is an equivalence relation.

Proof sketch

The following three properties need to be verified.

- **Reflexive**: Each $h \in G$ is conjugate to itself.
- **Symmetric**: If g is conjugate to h, then h is conjugate to g.
- **Transitive**: If g is conjugate to h, and h is conjugate to k, then g is conjugate to k.

As with any equivalence relation, the set is partitioned into equivalence classes.

The "class equation"

For any finite group G,

$$|G| = |Z(G)| + \sum \big| \mathrm{cl}_G(h_i) \big|,$$

where the sum is taken over distinct conjugacy classes of size greater than 1.

Conjugate elements

Proposition

Every normal subgroup is the union of conjugacy classes.

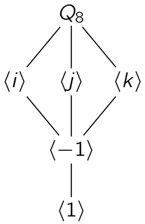
Proof

If $n \in N \trianglelefteq G$, then $xnx^{-1} \in xNx^{-1} = N$, and hence $\mathrm{cl}_G(n) \subseteq N$. □

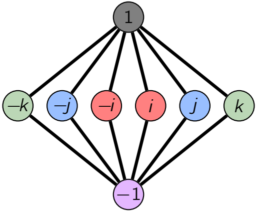
Let's look at Q_8, all of whose subgroups are normal.

- Since $i \notin Z(Q_8) = \{\pm 1\}$, we know $\left|\mathrm{cl}_{Q_8}(i)\right| > 1$.
- Also, $\langle i \rangle = \{\pm 1, \pm i\}$ is a union of conjugacy classes.
- Therefore $\mathrm{cl}_{Q_8}(i) = \{\pm i\}$.

Similarly, $\mathrm{cl}_{Q_8}(j) = \{\pm j\}$ and $\mathrm{cl}_{Q_8}(k) = \{\pm k\}$.



1	i	j	k
-1	$-i$	$-j$	$-k$



Conjugation preserves structure

Think back to linear algebra. Matrices A and B are similar (=conjugate) if $A = PBP^{-1}$.

Conjugate matrices have the same eigenvalues, trace, and determinant.

In fact, they represent the same linear map, but under a change of basis.

Central theme in mathematics

Two things that are conjugate have the same structure.

Let's start with a basic property preserved by conjugation.

Proposition

Conjugate elements in a group have the same order.

Proof

Consider h and $g = xhx^{-1}$. Suppose $|h| = n$, then

$$g^n = (xhx^{-1})^n = (xhx^{-1})(xhx^{-1}) \cdots (xhx^{-1}) = xh^n x^{-1} = xex^{-1} = e.$$

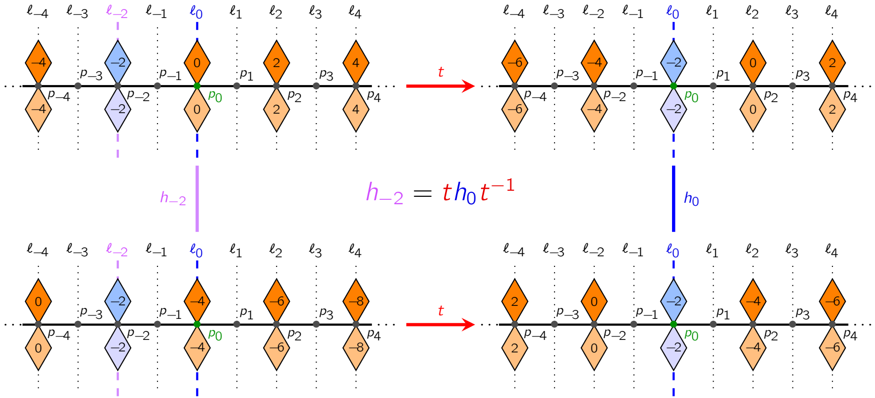
Therefore, $|g| = |xhx^{-1}| \leq |h|$. Reversing roles of g and h gives $|h| \leq |g|$. $\square$

Conjugation preserves structure

To understand what we mean by conjugation preserves structure, let's revisit frieze groups.

Let $h = h_0$ denote the reflection across the central axis, ℓ_0.

Suppose we want to reflect across a different axis, say ℓ_{-2}.



It should be clear that all reflections (resp., rotations) of the "same parity" are conjugate.

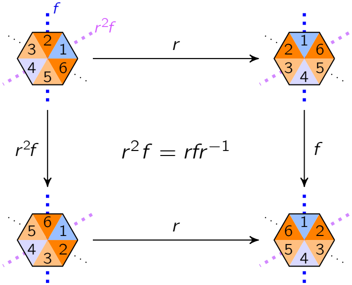
Conjugacy classes in D_n

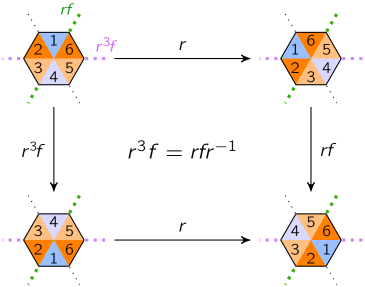
The dihedral group D_n is a "finite version" of a previous frieze group.

When n is even, there are two "*types of reflections*" of an n-gon:

1. $r^{2k}f$ is across an axis that bisects two sides
2. $r^{2k+1}f$ is across an axis that goes through two corners.

Here is a visual reason why each of these two types form a conjugacy class in D_n.





What do you think the conjugacy classes of a reflection is in D_n when n is odd?

Next, let's verify the conjugacy classes algebraically.

Conjugacy classes in D_6

Let's find the conjugacy classes of D_6 algebraically.

The center is $Z(D_6) = \{1, r^3\}$; these are the *only* elements in size-1 conjugacy classes.

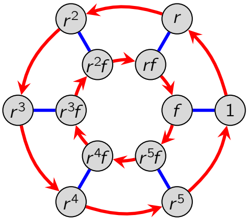
The only two elements of order 6 are r and r^5, so $\mathrm{cl}_{D_6}(r) = \{r, r^5\}$.

The only two elements of order 3 are r^2 and r^4, so $\mathrm{cl}_{D_6}(r^2) = \{r^2, r^4\}$.

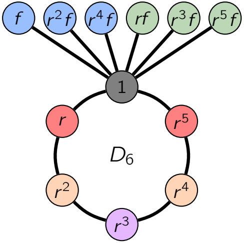
For a reflection $r^i f$, we need to consider two cases; conjugating by r^j and by $r^j f$:

- $r^j(r^i f)r^{-j} = r^j r^i r^j f = r^{i+2j} f$
- $(r^j f)(r^i f)(r^j f)^{-1} = (r^j f)(r^i f) f r^{-j} = r^j f r^{i-j} = r^j r^{j-i} f = r^{2j-i} f.$

Thus, $r^i f$ and $r^k f$ are conjugate iff i and k have the same parity.

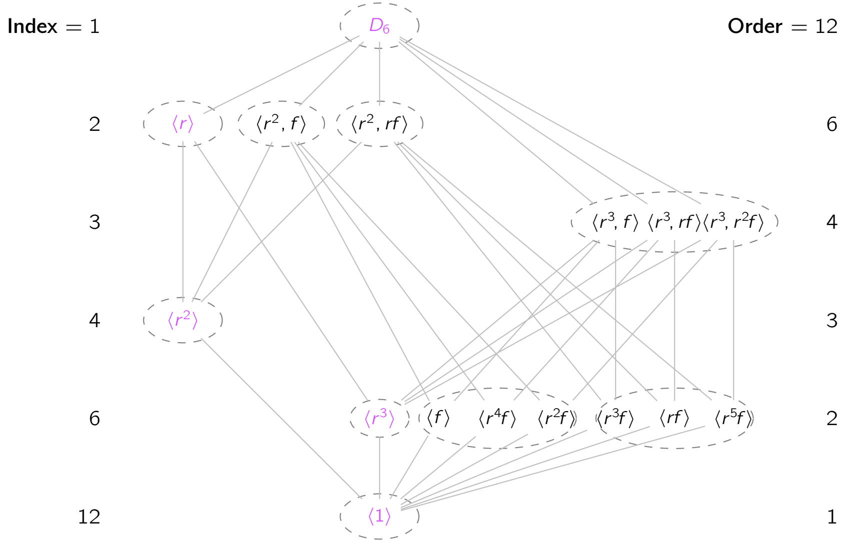


1	r	r^2	f	$r^2 f$	$r^4 f$
r^3	r^5	r^4	rf	$r^3 f$	$r^5 f$

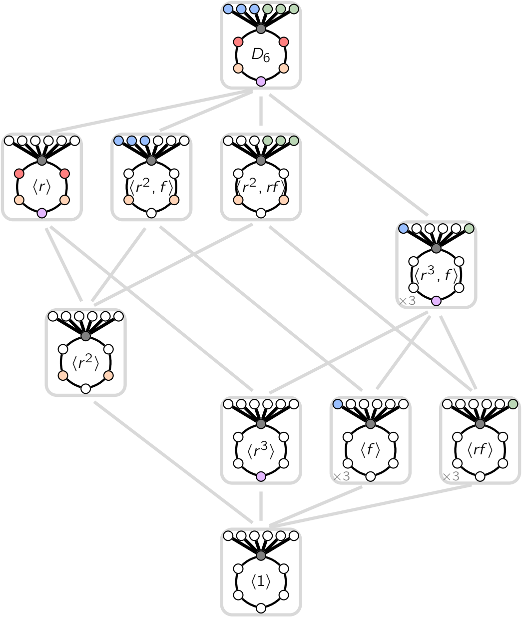


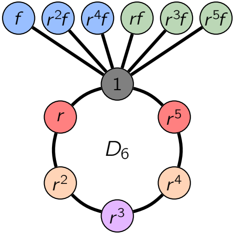
The subgroup lattice of D_6

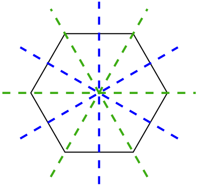
We can now deduce the conjugacy classes of the subgroups of D_6.



The subgroup diagram of D_6

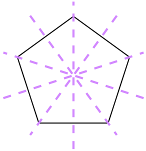


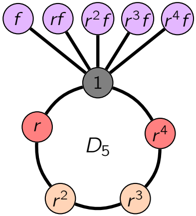


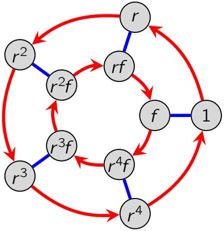


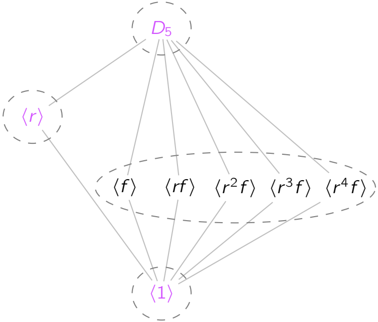
Conjugacy classes in D_5

Since $n = 5$ is odd, all reflections in D_5 are conjugate.









1	rf	r^3f	r	r^4
f	r^2f	r^4f	r^2	r^3

Cycle type and conjugacy in the symmetric group

We introduced **cycle type** in back in Chapter 2.

This is best seen by example. There are five cycle types in S_4:

example element	e	(12)	(234)	(1234)	(12)(34)
parity	even	odd	even	odd	even
# elts	1	6	8	6	3

Definition

Two elements in S_n have the same cycle type if when written as a product of disjoint cycles, there are the same number of length-k cycles for each k.

Theorem

Two elements $g, h \in S_n$ are conjugate if and only if they have the same cycle type.

For example, permutations in S_5 fall into seven cycle types (conjugacy classes):

$$\text{cl}(e), \quad \text{cl}((12)), \quad \text{cl}((123)), \quad \text{cl}((1234)), \quad \text{cl}((12345)), \quad \text{cl}((12)(34)), \quad \text{cl}((12)(345)).$$

Big idea

Conjugate permutations have the same structure: they are *the same up to renumbering*.

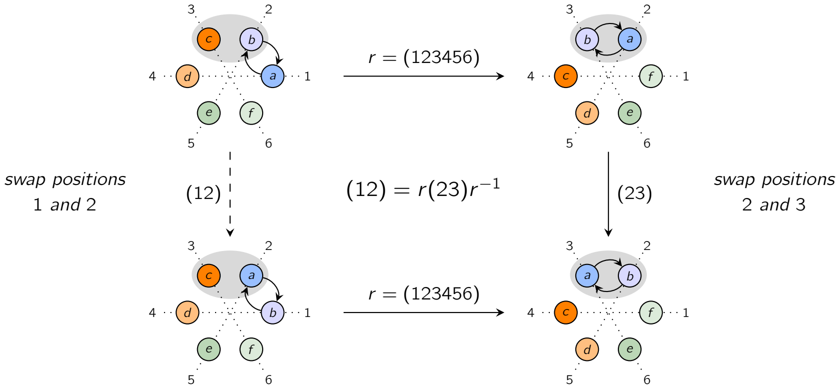
Conjugation preserves structure in the symmetric group

The symmetric group $G = S_6$ is generated by a transposition $(i\ i+1)$ and an n-cycle.

Consider the permutations of seating assignments around a circular table achievable by

- (23): "*people in chairs* 2 *and* 3 *may swap seats*"
- (123456): "*people may cyclically rotate seats counterclockwise*"

Here's how to get people in chairs 1 and 2 to swap seats:



The subgroup lattice of S_4

Exercise

Partition the subgroup lattice of S_4 into conjugacy classes by inspection alone.