Visual Algebra

Lecture 3.10: Centralizers

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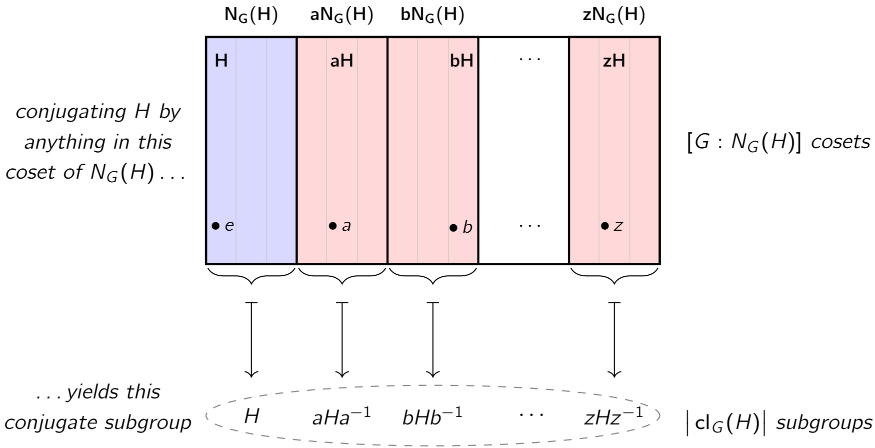
Theorem (Lecture 3.5)

For any subgroup $H \leq G$, the **size of its conjugacy class** is the **index of its normalizer**:

$$\big|\operatorname{cl}_G(H)\big| = [G : N_G(H)].$$



In this lecture, we'll see an analogue for **conjugacy classes of elements**.

Centralizers

Definition

The centralizer of a set $H \subseteq G$ is the set of elements that commute with everything in H:

$$C_G(H) = \big\{x \in G \mid xh = hx, \text{ for all } h \in H\big\} \leq G.$$

Usually, $H = \{h\}$ (not a group!), in which case we'll write $C_G(h)$.

Exercise: (i) $C_G(h)$ contains at least $\langle h \rangle$, (ii) if $xh = hx$, then $x\langle h \rangle \subseteq C_G(h)$.

Definition

Let $h \in G$ with $[G : \langle h \rangle] = n < \infty$. The **degree of centrality** of h is

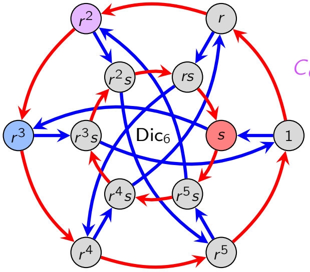
$$\text{Deg}_G^C(h) := \frac{|C_G(h)|}{|G|} = \frac{1}{[G : C_G(h)]} = \frac{\#\text{ elements } x \in G \text{ for which } xh = hx}{\#\text{ elements } x \in G}$$

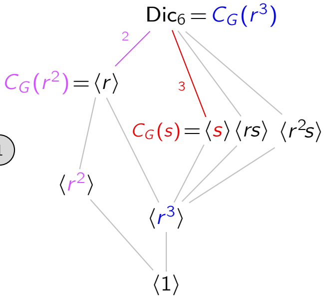
- If $\text{Deg}_G^C(h) = 1$, then h is central.
- If $\text{Deg}_G^C(h) = \frac{1}{n}$, we'll say h is fully uncentral.
- If $\frac{1}{n} < \text{Deg}_G^C(h) < 1$, we'll say h is moderately uncentral.

Big idea

The degree of centrality measures *how close to being central* an element is.

An example: element conjugacy classes and centralizers in Dic_6





rs	r^3s	r^5s
s	r^2s	r^4s
r^3	r^2	r^4
1	r	r^5

conjugacy classes

r^2	r^5	r^2s	r^5s
r	r^4	rs	r^4s
1	r^3	s	r^3s

$[G : C_G(r^3)] = 1$
"central"

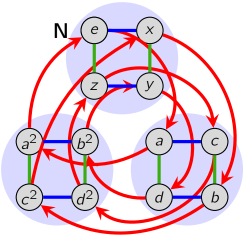
rs	r^3s	r^5s
s	r^2s	r^4s
r	r^3	r^5
1	r^2	r^4

$[G : C_G(r^2)] = 2$
"moderately uncentral"

r^2	r^2s	r^5	r^5s
r	rs	r^4	r^4s
1	s	r^3	r^3s

$[G : C_G(s)] = 3$
"fully unncentral"

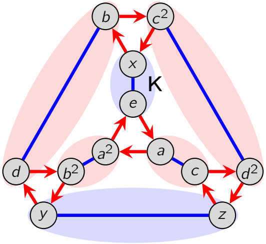
An old example: subgroup conjugacy classes and normalizers in A_4



(124)	(234)	(143)	(132)
(123)	(243)	(142)	(134)
e	(12)(34)	(13)(24)	(14)(23)

$[A_4 : N_{A_4}(N)] = 1$

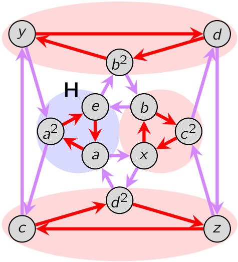
"normal"



(124)	(234)	(143)	(132)
(123)	(243)	(142)	(134)
e	(12)(34)	(13)(24)	(14)(23)

$[A_4 : N_{A_4}(K)] = 3$

"moderately unnormal"



(14)(23)	(142)	(143)
(13)(24)	(243)	(124)
(12)(34)	(134)	(234)
e	(123)	(132)

$[A_4 : N_{A_4}(H)] = 4$

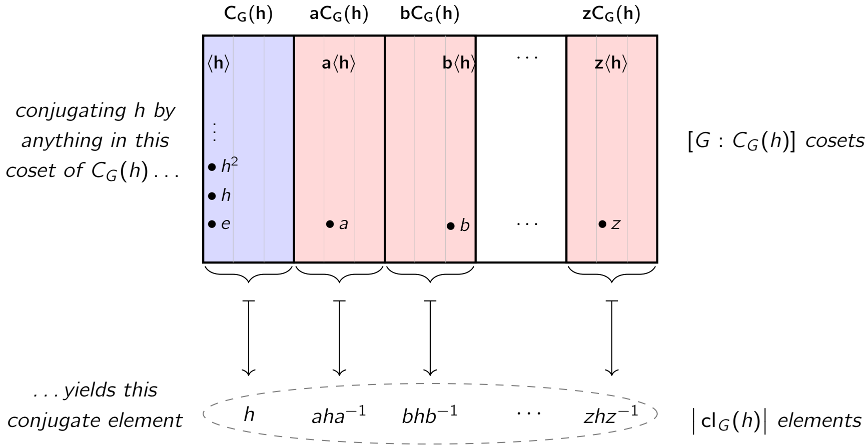
"fully unnormal"

The number of conjugate subgroups

Theorem

For any element $h \leq G$, the **size of its conjugacy class** is the **index of its centralizer**:

$$\big|\mathrm{cl}_G(h)\big| = [G : C_G(h)].$$



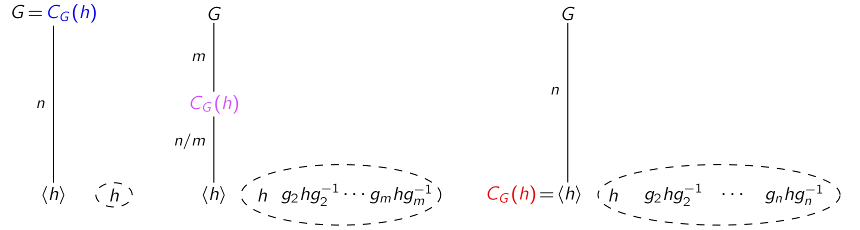
The number of conjugate elements

Theorem (restated)

Let $h \in G$ with $[G : \langle h \rangle] = n < \infty$. Then

$$\big|\mathrm{cl}_G(h)\big| = [G : C_G(h)] = \frac{\#\text{ elts } x \in G}{\#\text{ elts } x \in G \text{ for which } xh = hx} = \frac{1}{\mathrm{Deg}_G^C(h)}.$$

That is, there are exactly $[G : C_G(h)]$ elements conjugate to h.



central
$\big|\mathrm{cl}_G(h)\big| = 1$

moderately uncentral
$1 < \big|\mathrm{cl}_G(h)\big| < [G : \langle h \rangle]$

fully uncentral
$\big|\mathrm{cl}_G(h)\big| = [G : \langle h \rangle]$; as large as possible

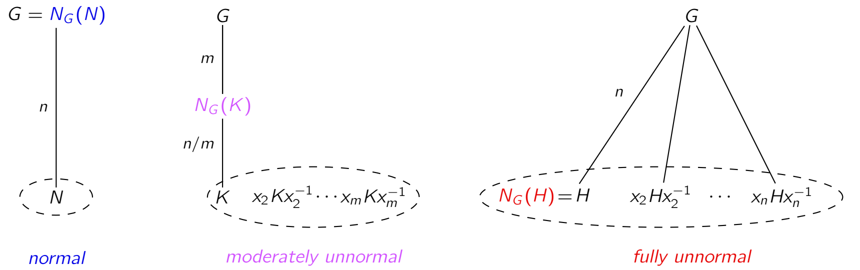
The number of conjugate subgroups

Theorem (Lecture 3.5)

Let $H \leq G$ with $[G:H] = n < \infty$. Then

$$\left|\mathrm{cl}_G(H)\right| = [G : N_G(H)] = \frac{\#\text{ elts } x \in G}{\#\text{ elts } x \in G \text{ for which } xH = Hx} = \frac{1}{\mathrm{Deg}_G^{\triangleleft}(H)}.$$

That is, H has exactly $[G : N_G(H)]$ conjugate subgroups.



$\left|\mathrm{cl}_G(N)\right| = 1$

$1 < \left|\mathrm{cl}_G(K)\right| < [G : K]$

$\left|\mathrm{cl}_G(H)\right| = [G : H]$; as large as possible

Conjugacy class size

Theorem (number of conjugate subgroups)

The conjugacy class of $H \leq G$ contains exactly $[G : N_G(H)]$ subgroups.

Proof (roadmap)

Construct a bijection between left cosets of $N_G(H)$ and conjugate subgroups of H:

"$xHx^{-1} = yHy^{-1}$ *iff* x *and* y *are in the same left coset of* $N_G(H)$."

Define $\phi \colon \{\text{left cosets of } N_G(H)\} \longrightarrow \{\text{conjugates of } H\}, \qquad \phi \colon xN_G(H) \longmapsto xHx^{-1}$.

Theorem (number of conjugate elements)

The conjugacy class of $h \in G$ contains exactly $[G : C_G(h)]$ elements.

Proof (roadmap)

Construct a bijection between left cosets of $C_G(h)$, and elements in $\mathrm{cl}_G(h)$:

"$xhx^{-1} = yhy^{-1}$ *iff* x and y are in the same left coset of $C_G(h)$."

Define $\phi \colon \{\text{left cosets of } C_G(h)\} \longrightarrow \{\text{conjugates of } h\}, \qquad \phi \colon xC_G(h) \longmapsto xhx^{-1}$.

Conjugacy class summary: subgroups vs. elements

The relationship between conjugacy classes and cosets of a certain subgroup are analogous.

	conjugacy of subgroups	conjugacy of elements
objects	$H \leq G$	$h \in G$
conjugacy class	$\mathrm{cl}_G(H)$	$\mathrm{cl}_G(h)$
they partition	the subgroups of G	the elements of G
-izer subgroup	Normalizer $N_G(H)$	Centralizer $C_G(h)$
$\big\lvert$conj. class$\big\rvert$	$[G : N_G(H)]$	$[G : C_G(h)]$
"best case" (size-1 class)	$N_G(H)=G$ (iff $H \trianglelefteq G$)	$C_G(h)=G$ (iff $h \in Z(G)$)
"worst case" (max'l class)	$N_G(H)=H$; "*fully unnormal*"	$C_G(h)=\langle h\rangle$; "*fully uncentral*"

Both are special cases of the orbit-stabilizer theorem, from Chapter 5 (group actions).