Visual Algebra

Lecture 4.6: Subquotients

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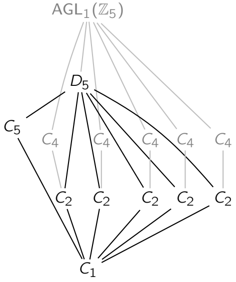
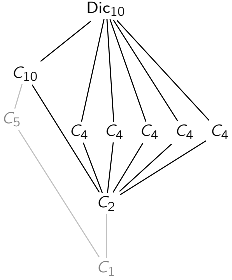
Summary of the isomorphism theorems

The isomorphism theorems

- Fundamental homomorphism theorem: "*All homomorphic images are quotients*"
- Correspondence theorem: Characterizes "*subgroups of quotients*"
- Fraction theorem: Characterizes "*quotients of quotients*"
- Diamond theorem: "*Duality of subquotients*."

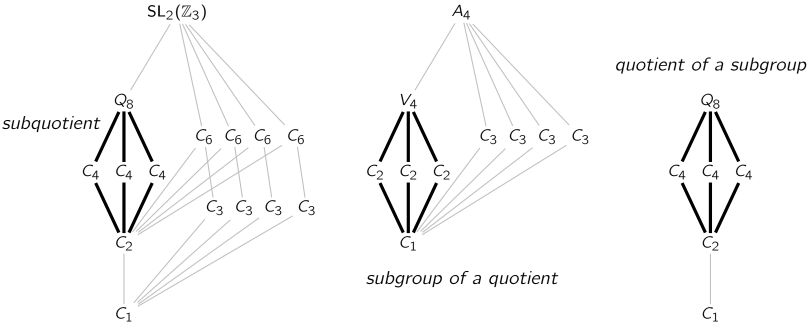




Subquotients

The isomorphism theorems

- Fundamental homomorphism theorem: "*All homomorphic images are quotients*"
- Correspondence theorem: Characterizes "*subgroups of quotients*"
- Fraction theorem: Characterizes "*quotients of quotients*"
- Diamond theorem: "*Duality of subquotients*."



The diamond theorem: duality of subquotients

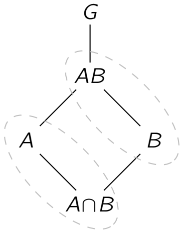
Diamond theorem

Suppose $A, B \leq G$, and that A normalizes B. Then

(i) $A \cap B \trianglelefteq A$ and $B \trianglelefteq AB$.

(ii) The following quotient groups are isomorphic:

$$AB/B \cong A/(A \cap B)$$



Proof (sketch)

Define the following map

If we can show:

$$\phi\colon A \longrightarrow AB/B\,, \qquad \phi\colon a \longmapsto aB\,.$$

1. ϕ is a homomorphism,
2. ϕ is surjective (onto),
3. $\text{Ker}(\phi) = A \cap B$,

then the result will follow *immediately* from the FHT. The details are left as an exercise.

Corollary

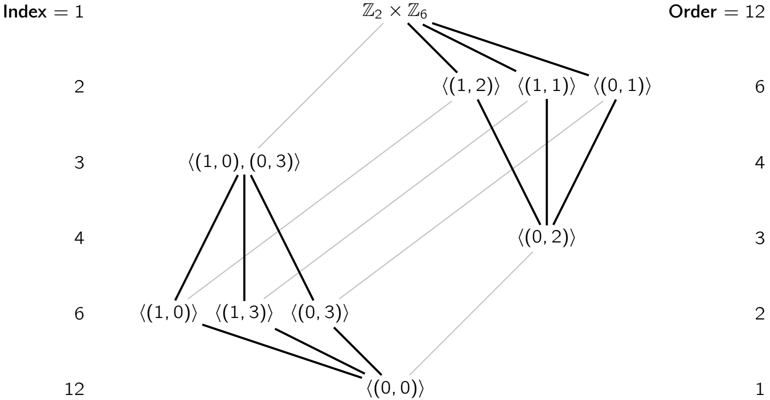
Let $A, B \leq G$, with one of them normalizing the other. Then $|AB| = \dfrac{|A| \cdot |B|}{|A \cap B|}$.

The diamond theorem: duality of subquotients

Let $G = \mathbb{Z}_2 \times \mathbb{Z}_6$, and consider subgroups $A = \langle (1,0), (0,3) \rangle$, and $B = \langle (0,2) \rangle$.

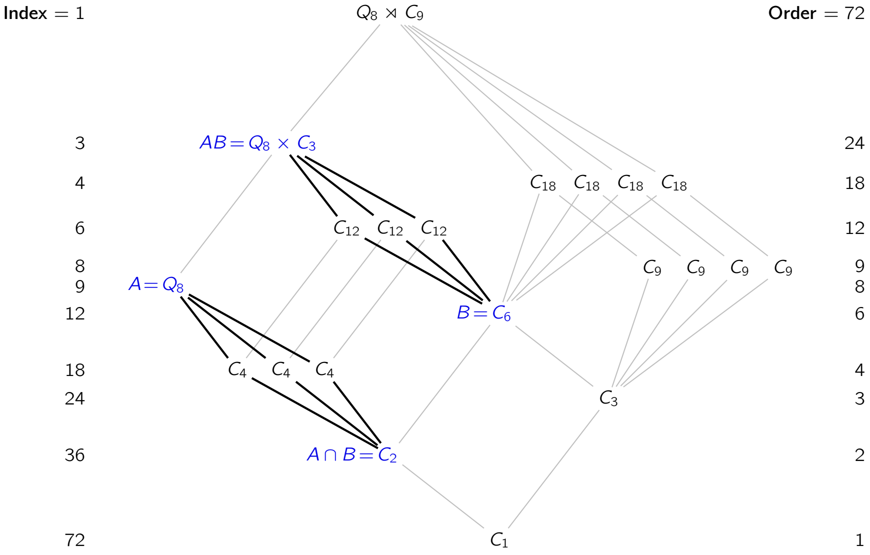
Then $G = AB$, and $A \cap B = \langle (0,0) \rangle$.

Let's interpret the diamond theorem $AB/B \cong A/A \cap B$ in terms of the subgroup lattice.



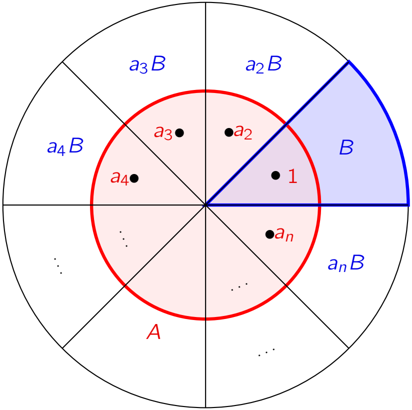
The fact that the subgroup lattice of V_4 is diamond shaped is coincidental.

The diamond theorem: duality of subquotients



The diamond theorem illustrated by a "pizza diagram"

The following analogy is due to Douglas Hofstadter:



$AB =$ large pizza

$A =$ small pizza

$B =$ large pizza slice

$A \cap B =$ small pizza slice

$AB/B = \{$large pizza slices$\}$

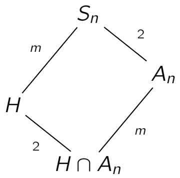
$A/(A \cap B) = \{$small pizza slices$\}$

Diamond theorem: $AB/B \cong A/(A \cap B)$

An application to permutation groups

Proposition

Suppose H is a subgroup of S_n that is not contained in A_n. Then exactly half of the permutations in H are even.



Index	**Order**
$= 1$	$= n!$
2	$n!/2$
m	$n!/m$
$2m$	$n!/2m$

Proof

It suffices to show that $[H : H \cap A_n] = 2$, or equivalently, that $H/(H \cap A_n) \cong C_2$.

Since $H \not\leq A_n$, the product HA_n must be strictly larger, and so $HA_n = S_n$.

By the diamond theorem,

$$H/(H \cap A_n) = HA_n/A_n = S_n/A_n \cong C_2.$$

□

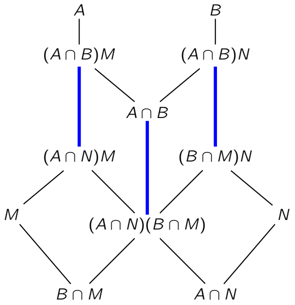
A theorem of Hans Zassenhaus

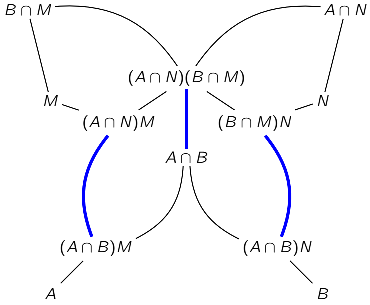
Butterfly lemma (see book for proof)

Let A, B be subgroups of a group, that contain $M \trianglelefteq A$ and $N \trianglelefteq B$. Then

1. $(A \cap N)M \trianglelefteq (A \cap B)M$,
2. $(B \cap M)N \trianglelefteq (A \cap B)N$,
3. The following quotient groups are isomorphic:

$$\frac{(A \cap B)M}{(A \cap N)M} \cong \frac{(A \cap B)N}{(B \cap M)N}.$$





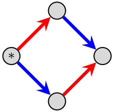
Commutators

We've seen how to divide $\mathbb{Z}$ by $\langle 12 \rangle$, thereby "forcing" all multiples of 12 to be zero. This is one way to construct the integers modulo 12: $\mathbb{Z}_{12} \cong \mathbb{Z}/\langle 12 \rangle$.

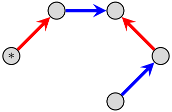
Now, suppose G is nonabelian. We'd like to divide G by its "non-abelian parts," making them zero and leaving only "abelian parts" in the resulting quotient.

A commutator is an element of the form $aba^{-1}b^{-1}$. Since G is nonabelian, *there are non-identity commutators:* $aba^{-1}b^{-1} \neq e$ *in* G.

$ab = ba$



$ab \neq ba$



In this case, the set $C := \{aba^{-1}b^{-1} \mid a, b \in G\}$ contains *more* than the identity.

Definition

The commutator subgroup G' of G is

$$G' := \langle aba^{-1}b^{-1} \mid a, b \in G \rangle .$$

The commutator subgroup is normal in G, and G/G' is abelian (exercise).

The abelianization of a group

Definition

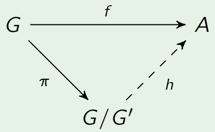
The abelianization of G is the quotient group G/G'.

The commutator subgroup G' is the smallest normal subgroup N of G such that G/N is abelian. [Note that G would be the "largest" such subgroup.]

Equivalently, the quotient G/G' is the largest abelian quotient of G. [Note that $G/G \cong \langle e \rangle$ would be the "smallest" such quotient.]

Universal property of commutator subgroups

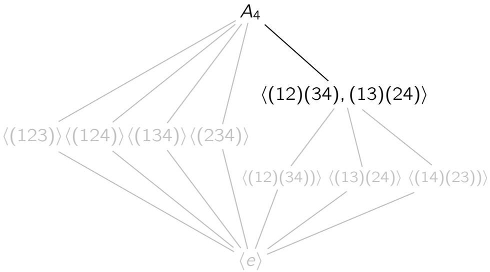
Suppose $f\colon G \to A$ is a homomorphism to an abelian group A. Then there is a unique homomorphism $h\colon G/G' \to A$ such that $f = h \circ \pi$:

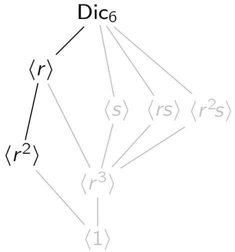


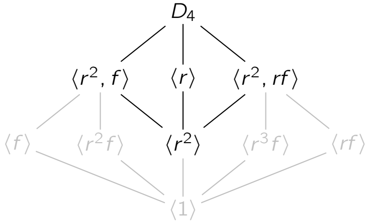
We say that f "factors through" the abelianization, G/G'.

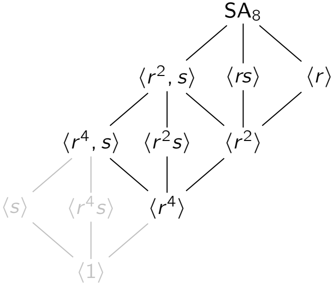
Some examples of abelianizations

By the isormophism theorems, we can usually identitfy the commutator subgroup G and abelianation by inspection, from the subgroup lattice.









Higher commutator subgroups

We can iterate the process of taking commutators.

We'll study the successive subquotients G/G', G'/G'', G''/G''', ... in Chapter 6.