Visual Algebra

Lecture 4.7: Automorphisms

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Automorphisms

We have already seen automorphisms of cyclic groups: "*structure-preserving rewirings*."

For a general group G, an automorphism is a isomorphism $\phi\colon G \to G$.

The set of automorphisms of G defines the automorphism group of G, denoted $\mathrm{Aut}(G)$.

Proposition

The automorphism group of $\mathbb{Z}_n$ is $\mathrm{Aut}(\mathbb{Z}_n) = \{\sigma_a \mid a \in U_n\} \cong U_n$, where

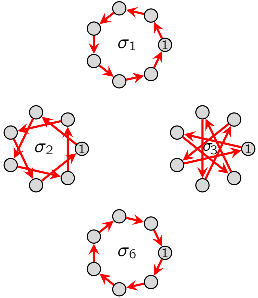
$$\sigma_a \colon \mathbb{Z}_n \longrightarrow \mathbb{Z}_n\,, \qquad \sigma_a(1) = a\,.$$

$\mathbf{U_7} = \langle \mathbf{3} \rangle \cong \mathbf{C_6}$

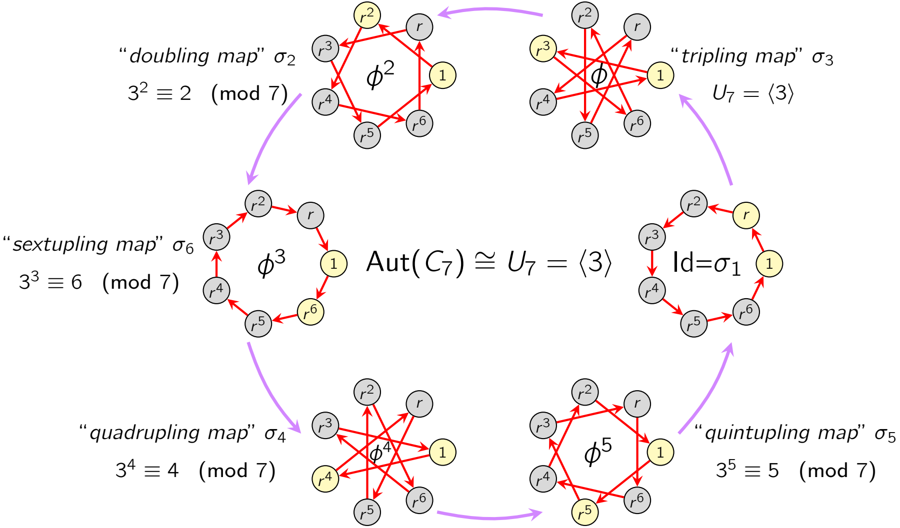
	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

$\mathrm{Aut}(\mathbf{C_7}) = \langle \sigma_3 \rangle \cong \mathbf{U_7}$

	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6
σ_1	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6
σ_2	σ_2	σ_4	σ_6	σ_1	σ_3	σ_5
σ_3	σ_3	σ_6	σ_2	σ_5	σ_1	σ_4
σ_4	σ_4	σ_1	σ_5	σ_2	σ_6	σ_3
σ_5	σ_5	σ_3	σ_1	σ_6	σ_4	σ_2
σ_6	σ_6	σ_5	σ_4	σ_3	σ_2	σ_1



An example: the automorphism group of C_7



Automorphisms of noncyclic groups

An automorphism is determined by where it sends the generators.

Examples

1. An automorphism ϕ of $V_4 = \langle h, v \rangle$ is determined by the image of h and v.

   There are 3 choices for $\phi(h)$, then 2 choices for $\phi(v)$, thus $|\operatorname{Aut}(V_4)| = 6$.

   Every permutation of $\{h, v, r\}$ is an automorphism, and so $\operatorname{Aut}(V_4) \cong S_3$.

2. Every $\phi \in \operatorname{Aut}(D_3)$ is determined by $\phi(r)$ and $\phi(f)$.

   Since automorphisms preserve order, if $\phi \in \operatorname{Aut}(D_3)$, then

$$\phi(1) = 1\,, \qquad \phi(r) = \underbrace{r \text{ or } r^2}_{\text{2 choices}}, \qquad \phi(f) = \underbrace{f,\ rf, \text{ or } r^2 f}_{\text{3 choices}}\,.$$

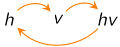
   Thus, $|\operatorname{Aut}(D_3)| \leq 6$. Both of the following define automorphisms of D_3:

$$\begin{cases} \alpha(r) = r \\ \alpha(f) = rf \end{cases} \qquad\qquad \begin{cases} \beta(r) = r^2 \\ \beta(f) = f \end{cases}$$

It is elementary to check that $\alpha\beta = \beta\alpha^2$, and so $\operatorname{Aut}(D_3) \cong D_3 \cong S_3$.

Automorphisms of $V_4 = \langle h, v \rangle$

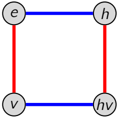
The following permutations are both automorphisms:

$\boldsymbol{\alpha}$: 

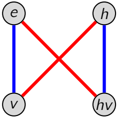
and

β : 

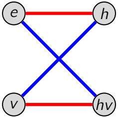
$h \overset{id}{\longmapsto} h$
$v \longmapsto v$
$hv \longmapsto hv$



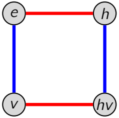
$h \overset{\alpha}{\longmapsto} v$
$v \longmapsto hv$
$hv \longmapsto h$



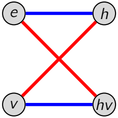
$h \overset{\alpha^2}{\longmapsto} hv$
$v \longmapsto h$
$hv \longmapsto v$



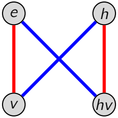
$h \overset{\beta}{\longmapsto} v$
$v \longmapsto h$
$hv \longmapsto hv$



$h \overset{\alpha\beta}{\longmapsto} h$
$v \longmapsto hv$
$hv \longmapsto v$



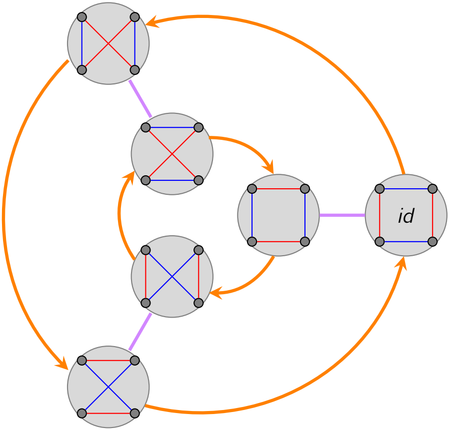
$h \overset{\alpha^2\beta}{\longmapsto} hv$
$v \longmapsto v$
$hv \longmapsto h$



Automorphisms of $V_4 = \langle h, v \rangle$

Here is the Cayley table and Cayley graph of $\mathbf{Aut}(V_4) = \langle \alpha, \beta \rangle \cong S_3 \cong D_3$.

	id	α	α^2	β	$\alpha\beta$	$\alpha^2\beta$
id	id	α	α^2	β	$\alpha\beta$	$\alpha^2\beta$
α	α	α^2	id	$\alpha\beta$	$\alpha^2\beta$	β
α^2	α^2	id	α	$\alpha^2\beta$	β	$\alpha\beta$
β	β	$\alpha^2\beta$	$\alpha\beta$	id	α^2	α
$\alpha\beta$	$\alpha\beta$	β	$\alpha^2\beta$	α	id	α^2
$\alpha^2\beta$	$\alpha^2\beta$	$\alpha\beta$	β	α^2	α	id



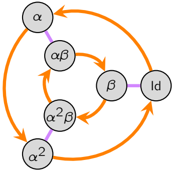
Recall that α and β can be thought of as the permutations 

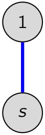
and 

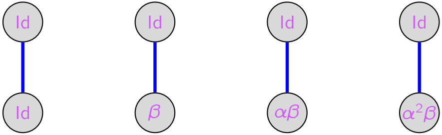
and so $\mathbf{Aut}(G) \hookrightarrow \mathbf{Perm}(G) \cong S_n$ always holds.

The construction of $V_4 \rtimes C_2$

A labeling map $\theta_i \colon C_2 \longrightarrow \mathrm{Aut}(V_4) \cong D_3$ is just a homomorphism. There are four:



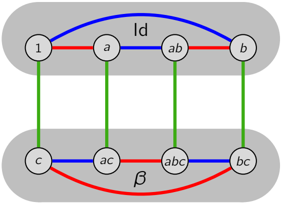




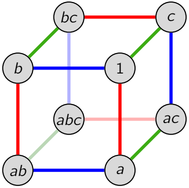
$s \overset{\theta}{\longmapsto} \mathrm{Id}$ $\quad s \overset{\theta_0}{\longmapsto} \beta$ $\quad s \overset{\theta_1}{\longmapsto} \alpha\beta$ $\quad s \overset{\theta_2}{\longmapsto} \alpha^2\beta$

Let's now carry out our "inflation method" to construct $V_4 \rtimes C_2$.

Start with a copy of $B = C_2$



Inflate each node, insert rewired versions of $A = V_4$, and connect corresponding nodes



rearrange the Cayley graph

What familiar group is $V_4 \rtimes C_2$?

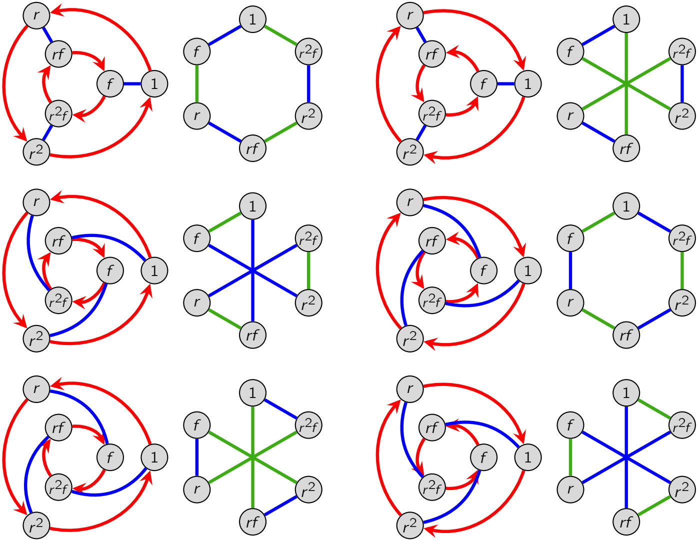
Automorphisms of D_3

$\boldsymbol{\alpha}$: r r^2 f rf r^2f and $\boldsymbol{\beta}$: r r^2 f rf r^2f

$r \xrightarrow{id} r$
$f \longrightarrow f$

$r \xrightarrow{\alpha} r$
$f \longrightarrow rf$

$r \xrightarrow{\alpha^2} r$
$f \longrightarrow r^2f$



$r \xrightarrow{\beta} r^2$
$f \longrightarrow f$

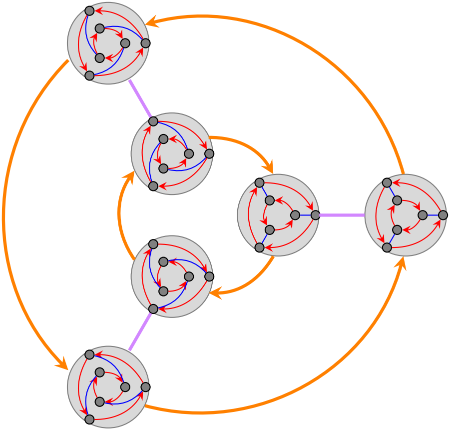
$r \xrightarrow{\alpha\beta} r^2$
$f \longrightarrow r^2f$

$r \xrightarrow{\alpha^2\beta} r^2$
$f \longrightarrow rf$

Automorphisms of D_3

Here is the Cayley table and Cayley graph of $\mathbf{Aut}(D_3) = \langle \alpha, \beta \rangle$.

	id	α	α^2	β	$\alpha\beta$	$\alpha^2\beta$
id	id	α	α^2	β	$\alpha\beta$	$\alpha^2\beta$
α	α	α^2	id	$\alpha\beta$	$\alpha^2\beta$	β
α^2	α^2	id	α	$\alpha^2\beta$	β	$\alpha\beta$
β	β	$\alpha^2\beta$	$\alpha\beta$	id	α^2	α
$\alpha\beta$	$\alpha\beta$	β	$\alpha^2\beta$	α	id	α^2
$\alpha^2\beta$	$\alpha^2\beta$	$\alpha\beta$	β	α^2	α	id

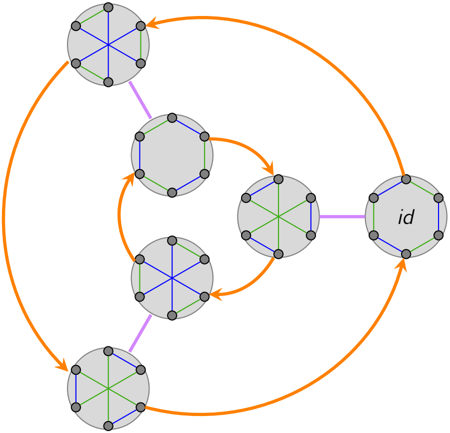


$\boldsymbol{\alpha}$: r $\quad r^2$ $\quad f$ $\quad rf$ $\quad r^2f$ $\qquad$ *and* $\qquad$ $\boldsymbol{\beta}$: r $\quad r^2$ $\quad f$ $\quad rf$ $\quad r^2f$

Automorphisms of D_3

Here is the Cayley table and Cayley graph of $\mathbf{Aut}(D_3) = \langle \alpha, \beta \rangle$.

	id	α	α^2	β	$\alpha\beta$	$\alpha^2\beta$
id	id	α	α^2	β	$\alpha\beta$	$\alpha^2\beta$
α	α	α^2	id	$\alpha\beta$	$\alpha^2\beta$	β
α^2	α^2	id	α	$\alpha^2\beta$	β	$\alpha\beta$
β	β	$\alpha^2\beta$	$\alpha\beta$	id	α^2	α
$\alpha\beta$	$\alpha\beta$	β	$\alpha^2\beta$	α	id	α^2
$\alpha^2\beta$	$\alpha^2\beta$	$\alpha\beta$	β	α^2	α	id



$\boldsymbol{\alpha}$: r $\quad r^2$ $\quad f$ $\quad rf$ $\quad r^2f$ *and* $\boldsymbol{\beta}$: r $\quad r^2$ $\quad f$ $\quad rf$ $\quad r^2f$

A few more examples of semidirect products

What groups are these?