# Visual Algebra

## Lecture 4.8: Inner and outer automorphisms

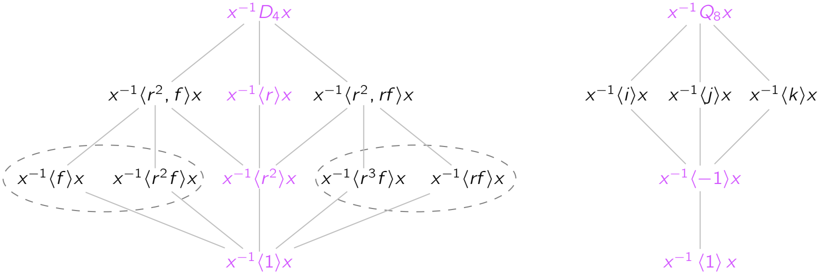
**Dr. Matthew Macauley**

School of Mathematical & Statistical Sciences
Clemson University
South Carolina, USA
http://www.math.clemson.edu/~macaule/

# Inner and outer automorphisms

Earlier in this class, we conjugated an entire group $G$ by a fixed element $x \in G$.

This is an example of an inner automorphism. Here are two examples:




This permutes subgroups *within a conjugacy class*: $r^{-1}\langle f\rangle r = \langle r^2f\rangle$.

Every subgroup of $Q_8$ is normal, thus any inner automorphism fixes every subgroup.

However, there is an automorphism of $Q_8$ that permutes subgroups, defined by

$$\phi\colon Q_8 \longrightarrow Q_8, \qquad \phi(i) = j, \quad \phi(j) = k \quad \Rightarrow \quad \phi(k) = \phi(ij) = \phi(i)\phi(j) = jk = i.$$

This is called an outer automorphism.

# The inner automorphism group

## Definition

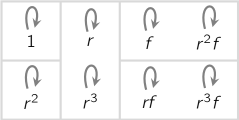
An inner automorphism of $G$ is an automorphism $\varphi_x \in \mathrm{Aut}(G)$ defined by

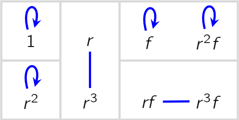
$$\varphi_x(g) := x^{-1}gx, \qquad \text{for some } x \in G.$$

The inner automorphisms of $G$ form a group, denoted $\mathrm{Inn}(G)$. (Exercise)

There are four inner automorphisms of $D_4$:

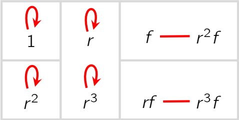
$\mathrm{Id} = \varphi_1 = \varphi_{r^2}$

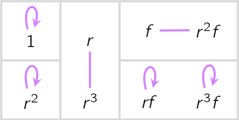



$\varphi_f = \varphi_{r^2 f}$

$\varphi_r = \varphi_{r^3}$




$\varphi_{rf} = \varphi_{r^3 f}$

Since $\varphi_x^2 = \mathrm{Id}$ for all of these, $\mathrm{Inn}(D_4) = \langle \varphi_r, \varphi_f \rangle \cong V_4$.

*Are there any other automorphisms of $D_4$?*

# The inner automorphism group

## Proposition (exercise)

$\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.

## Remarks

- Many books define $\varphi_x(g) = xgx^{-1}$. Our choice is so $\varphi_{xy} = \varphi_x\varphi_y$ (reading L-to-R).
- If $z \in Z(G)$, then $\varphi_z \in \mathrm{Inn}(G)$ is trivial.
- If $x = yz$ for some $z \in Z(G)$, then $\varphi_x = \varphi_y$ in $\mathrm{Inn}(G)$:

$$\varphi_x(g) = x^{-1}gx = (yz)^{-1}g(yz) = z^{-1}(y^{-1}gy)z = y^{-1}gy = \varphi_y(g).$$

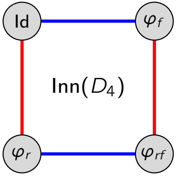
That is, if $x$ and $y$ are in the same coset of $Z(G)$, then $\varphi_x = \varphi_y$. (And conversely.)

| $Z$ | $rZ$ | $fZ$ | $rfZ$ |
|---|---|---|---|
| $1$ | $r$ | $f$ | $rf$ |
| $r^2$ | $r^3$ | $r^2f$ | $r^3f$ |

cosets of $Z(D_4)$ are in bijection with inner automorphisms of $D_4$

| | | | | |
|---|---|---|---|---|
| cl(1) | $1$ | $r$ | $f$ | $rf$ |
| cl($r^2$) | $r^2$ | $r^3$ | $r^2f$ | $r^3f$ |
| | | cl($r$) | cl($f$) | cl($rf$) |

inner automorphisms of $D_4$ permute elements within conjugacy classes



# The inner automorphism group

## Key point

Two elements $x, y \in G$ are in the same coset of $Z(G)$ if and only if $\varphi_x = \varphi_y$ in $\mathrm{Inn}(G)$.

## Proposition

In any group $G$, we have $G/Z(G) \cong \mathrm{Inn}(G)$.

## Proof

Consider the map

$$f \colon G \longrightarrow \mathrm{Inn}(G), \qquad x \longmapsto \varphi_x,$$

It is straightfoward to check this this is (i) a homomorphism, (ii) onto, and (iii) that $\mathrm{Ker}(f) = Z(G)$.

The result is now immediate from the FHT. □

We just saw that $\mathrm{Aut}(D_3) \cong D_3$, and we know that $Z(D_3) = \langle 1 \rangle$. Therefore,

$$\mathrm{Inn}(D_3) \cong D_3/Z(D_3) \cong D_3 \cong \mathrm{Aut}(D_3),$$

i.e., every automorphism is inner.

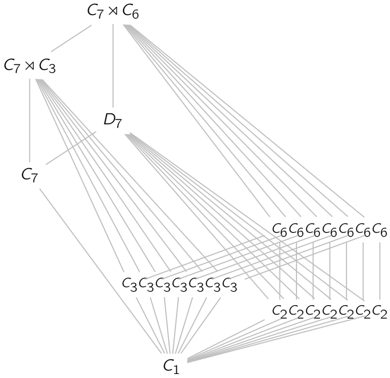
# Inn($G$) can *never* be a nontrivial cyclic subgroup

## Lemma

If $\text{Inn}(G) \cong G/Z(G)$ is cyclic, then $G$ is abelian.

$$\mathbf{G/Z(G)} = \langle \mathbf{gZ} \rangle, \textit{ where } Z = Z(G)$$

| | | | | | |
|---|---|---|---|---|---|
| $\bullet\, g^{n-1}$ | $\bullet\, g^{n-1}z_1$ | $\bullet\, g^{n-1}z_2$ | $\bullet\, g^{n-1}z_3$ | $\cdots$ | $\mathbf{g^{n-1}Z}$ |
| | | $\vdots$ | | | |
| $\bullet\, g^2$ | $\bullet\, g^2z_1$ | $\bullet\, g^2z_2$ | $\bullet\, g^2z_3$ | $\cdots$ | $\mathbf{g^2Z}$ |
| $\bullet\, g$ | $\bullet\, gz_1$ | $\bullet\, gz_2$ | $\bullet\, gz_3$ | $\cdots$ | $\mathbf{gZ}$ |
| $\bullet\, e$ | $\bullet\, z_1$ | $\bullet\, z_2$ | $\bullet\, z_3$ | $\cdots$ | $\mathbf{Z}$ |




If $G$ is abelian, then $Z(G) = G$.

## Corollary

For any group $G$, finite or infinite, $[G : Z(G)] \geq 4$.

# Inner automorphisms of $D_3$

Let's label each $\phi \in \mathrm{Aut}(D_3)$ with the corresponding inner automorphism.

$r \xrightarrow{Id} r$
$f \longrightarrow f$

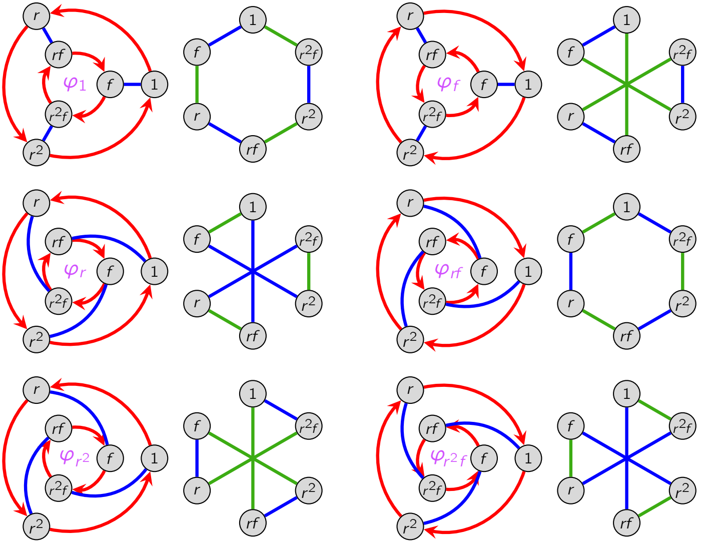
$r \xrightarrow{\beta} r^2$
$f \longrightarrow f$

$r \xrightarrow{\alpha} r$
$f \longrightarrow rf$

$r \xrightarrow{\alpha\beta} r^2$
$f \longrightarrow r^2 f$

$r \xrightarrow{\alpha^2} r$
$f \longrightarrow r^2 f$

$r \xrightarrow{\alpha^2\beta} r^2$
$f \longrightarrow rf$



# Automorphisms of $D_4$

Every automorphism of $D_4 = \langle r, f \rangle$ is determined by where it sends the generators:

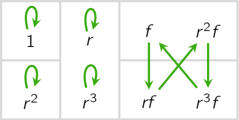
$$\phi(r) = \underbrace{r \text{ or } r^3}_{\text{2 choices}}, \qquad \phi(f) = \underbrace{f,\ rf,\ r^2f,\ r^3f, \text{ or } r^2}_{\text{5 choices}}.$$

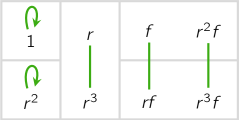
Thus $|\,\mathrm{Aut}(D_4)| \leq 10$. But $\mathrm{Inn}(D_4) \leq \mathrm{Aut}(D_4)$, forces $|\,\mathrm{Aut}(D_4)| = 4$ or $8$. Moreover,

$$\omega\colon D_4 \longrightarrow D_4, \qquad \omega(r) = r, \quad \omega(f) = rf$$

is an (outer) automorphism, which swaps the "two types" of reflections of the square.

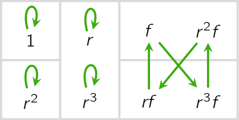
$\omega$

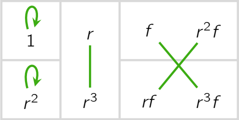



$\varphi_f \omega$

$\varphi_r \omega$

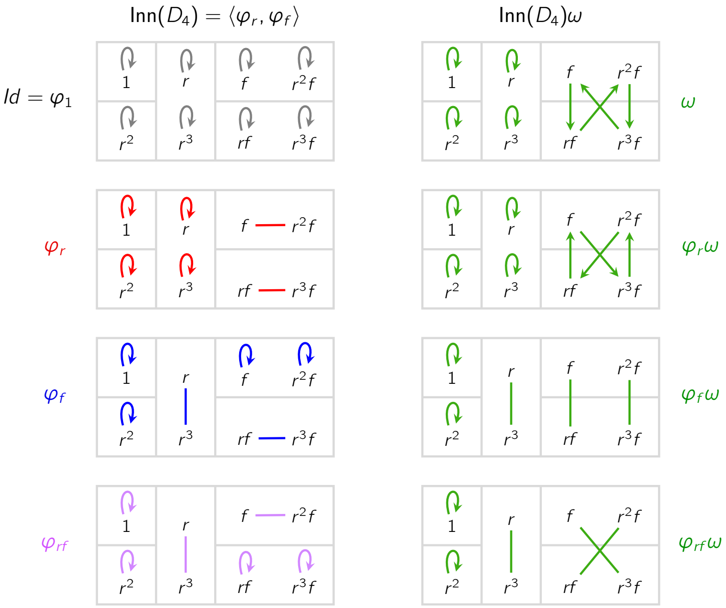



$\varphi_{rf} \omega$

$$\mathrm{Aut}(D_4) = \left\{ Id,\ \varphi_r,\ \varphi_f,\ \varphi_{rf},\ \omega,\ \varphi_r\omega,\ \varphi_f\omega,\ \varphi_{rf}\omega \right\} = \mathrm{Inn}(D_4) \cup \mathrm{Inn}(D_4)\omega \cong D_4.$$

# The full automorphism group of $D_4$

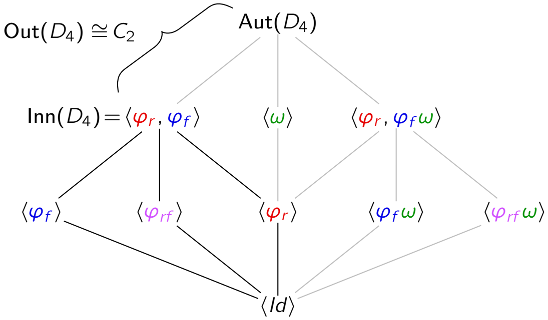


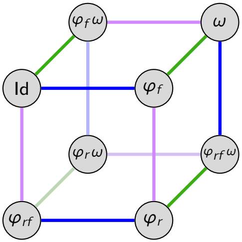
# The outer automorphism group

## Definition

An outer automorphism of $G$ is any automorphism that is not inner.

The outer automorphism group of $G$ is the quotient $\mathrm{Out}(G) := \mathrm{Aut}(G)/\mathrm{Inn}(G)$.




$\mathrm{Aut}(D_4) \cong \mathrm{Inn}(D_4) \rtimes \mathrm{Out}(D_4)$

Note that there are four outer automorphisms, but $|\mathrm{Out}(D_4)| = 2$.

We have seen: $\mathrm{Out}(V_4) \cong D_3$, $\mathrm{Out}(D_3) \cong \{\mathrm{Id}\}$, $\mathrm{Out}(D_4) \cong C_2$, $\mathrm{Out}(Q_8) \cong S_3$.

# Class automorphisms

## Proposition (exercise)

Automorphisms permute conjugacy classes. That is, $g, h \in G$ are conjugate if and only if $\phi(g)$ and $\phi(h)$ are conjugate.

It is natural to ask if an automorphism being inner is equivalent to being the identity permutation on conjugacy classes.

In other words:

"*if* $\phi \in \mathrm{Aut}(G)$ *sends every element to a conjugate, must* $\phi \in \mathrm{Inn}(G)$*?*"

The answer is "no". Burnside found examples of groups of order at least 729 that admit such an automorphism.

## Definition

A class automorphism is an automorphism that sends every element to another in its conjugacy class.

In 1947, G.E. Wall found a group of order 32 with a class automorphism that is outer.

# "A wrinkle in the mathematical universe" –John Baez

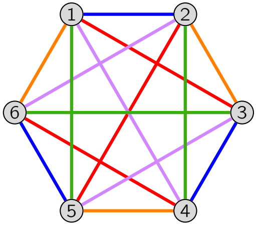
## Theorem

The outer automorphism group of $S_n$ is $\mathbf{Out}(S_n) \cong \begin{cases} C_2 & \text{if } n = 6 \\ C_1 & \text{otherwise} \end{cases}$

$S_6$ has an automorphism that permutes the following conjugacy classes:

$$\text{cl}_{S_6}((12)) \longleftrightarrow \text{cl}_{S_6}((12)(34)(56)), \qquad \text{cl}_{S_6}((123)) \longleftrightarrow \text{cl}_{S_6}((145)(256))$$

$$\text{cl}_{S_6}((12)(345)) \longleftrightarrow \text{cl}_{S_6}((123456))$$




$(12)(36)(45)$: swaps purple and red

$(13654)$: cycles blue $\to$ orange $\to$ purple $\to$ red $\to$ green

$S_5 \cong \langle (12)(36)(45),\ (13654) \rangle$

# An outer-automorphism of $S_6$

Image credit: Greg Egan