Visual Algebra

Lecture 4.9: External products and holomorphs

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Semidirect products, algebraically

Thus far, we've seen how to construct $A \rtimes_\theta B$ with our "inflation method."

Given A (for "*automorphism*") and B (for "*balloon*"), we label each inflated node $b \in B$ with $\phi \in \mathrm{Aut}(A)$ via some labeling map

$$\theta \colon B \longrightarrow \mathrm{Aut}(A).$$

Naturally, this can be defined algebraically. Denote multiplication in $A \times B$ by

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1 a_2, b_1 b_2).$$

Definition

The (external) **semidirect product** $A \rtimes_\theta B$ of A and B, with respect to the homomorphism

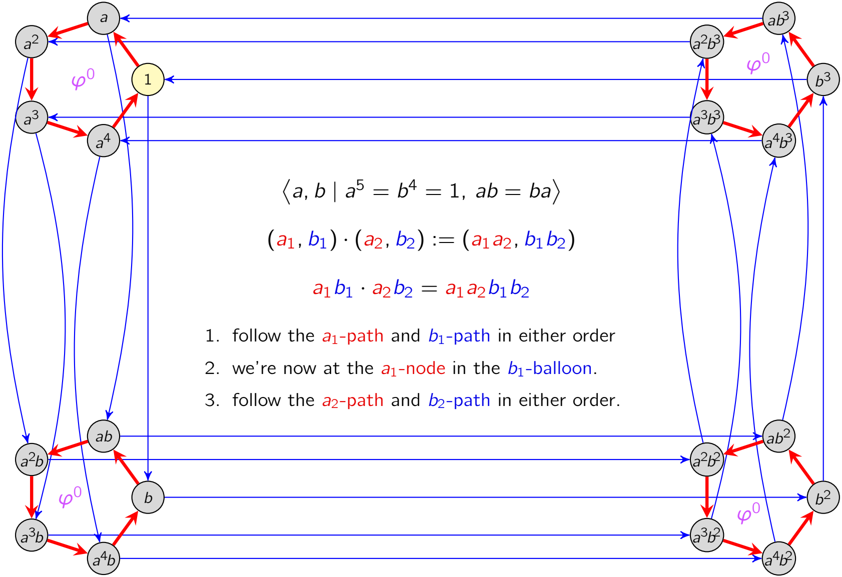
$$\theta \colon B \longrightarrow \mathrm{Aut}(A),$$

is on the underlying set $A \times B$, where the binary operation $*$ is defined as

$$(a_1, b_1) * (a_2, b_2) := (a_1, b_1) \cdot (\theta(b_1) a_2, b_2) = (a_1 \theta(b_1) a_2, b_1 b_2).$$

The isomorphic group on $B \times A$ by swapping the coordinates above is written $B \ltimes_\theta A$.

An example: the direct product $C_5 \times C_4$



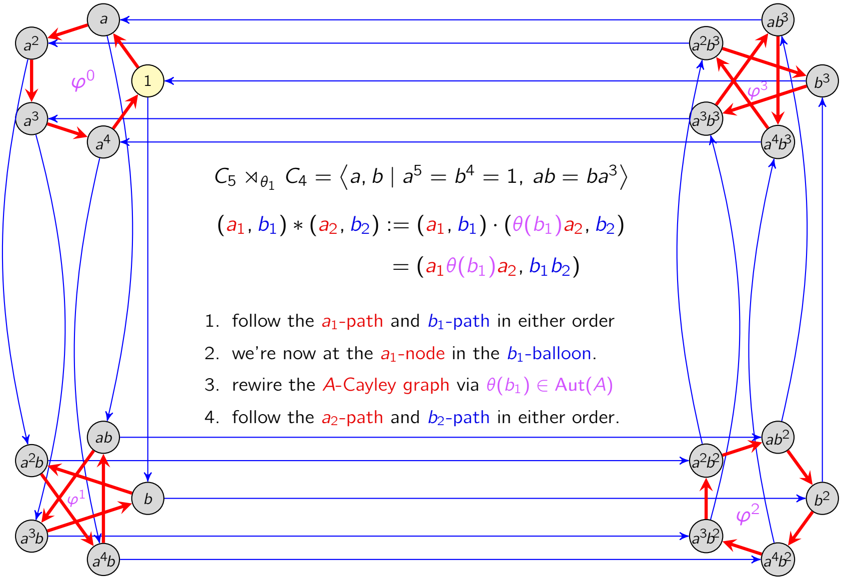
$$\langle a, b \mid a^5 = b^4 = 1,\ ab = ba \rangle$$

$$(a_1, b_1) \cdot (a_2, b_2) := (a_1 a_2, b_1 b_2)$$

$$a_1 b_1 \cdot a_2 b_2 = a_1 a_2 b_1 b_2$$

1. follow the a_1-path and b_1-path in either order
2. we're now at the a_1-node in the b_1-balloon.
3. follow the a_2-path and b_2-path in either order.

An example: the semidirect product $C_5 \rtimes_\theta C_4$



$$C_5 \rtimes_{\theta_1} C_4 = \langle a, b \mid a^5 = b^4 = 1,\ ab = ba^3 \rangle$$

$$(a_1, b_1) * (a_2, b_2) := (a_1, b_1) \cdot (\theta(b_1)a_2, b_2)$$

$$= (a_1\theta(b_1)a_2, b_1b_2)$$

1. follow the a_1-path and b_1-path in either order
2. we're now at the a_1-node in the b_1-balloon.
3. rewire the A-Cayley graph via $\theta(b_1) \in \mathrm{Aut}(A)$
4. follow the a_2-path and b_2-path in either order.

External semidirect products

Recall how to multipy in $A \rtimes_\theta B$:

$$(a_1, b_1) * (a_2, b_2) := (a_1, b_1) \cdot (\theta(b_1)a_2, b_2) = (a_1 \cdot \theta(b_1)a_2, b_1 b_2).$$

Lemma

The subgroup $A \times \{1\}$ is normal in $A \rtimes_\theta B$.

Proof

Let's conjugate an arbitrary element $(x, 1) \in A \times \{1\}$ by an element $(a, b) \in A \rtimes_\theta B$.

$$(a, b) * (x, 1) * (a, b)^{-1} = \big(a \cdot \underbrace{\theta(b)x}_{\in A}, b\big) * (a^{-1}, b^{-1}) = \big(\underbrace{a \cdot \theta(b)x \cdot \theta(b)a^{-1}}_{\in A}, 1\big) \in A \times \{1\}.$$

Not all books use the same notation for semidirect products. Ours is motivated by:

- In $A \times B$, both factors are normal (technically, $A \times \{1\}$ and $\{1\} \times B$).
- In $A \rtimes B$, the group on the "open" side of $\rtimes$ is normal.

The holomorph of a group

Definition

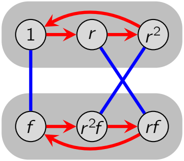
The holomorph of a group is $\mathrm{Hol}(G) = G \rtimes \mathrm{Aut}(G)$, with labeling map

$$\theta\colon \underbrace{\mathrm{Aut}(G)}_{B} \longrightarrow \underbrace{\mathrm{Aut}(G)}_{\mathrm{Aut}(A)}, \qquad \theta(\varphi) = \varphi.$$

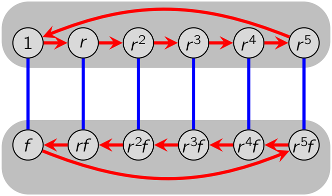
This means that the binary operation is

$$(g_1, \alpha) * (g_2, \beta) = (g_1 \cdot \alpha(g_2), \alpha\beta)$$

$\mathrm{Hol}(C_3) = C_3 \rtimes C_2$



$\mathrm{Hol}(C_6) = C_6 \rtimes C_2$

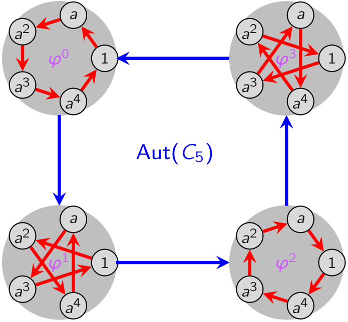


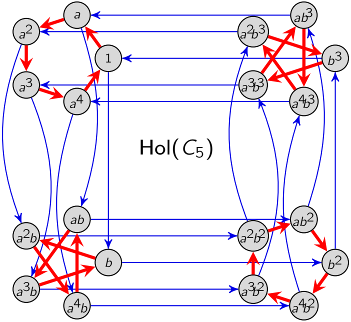
The holomorph of C_5

Recall that $\mathbf{Aut}(C_5) \cong C_4$. Thus, the holomorph of C_5 is

$$\mathbf{Hol}(C_5) \cong C_5 \rtimes \mathbf{Aut}(C_5) \cong C_5 \rtimes C_4.$$

We've already seen this construction.





This is the affine general linear group $\mathbf{AGL}_2(\mathbb{Z}_5)$.

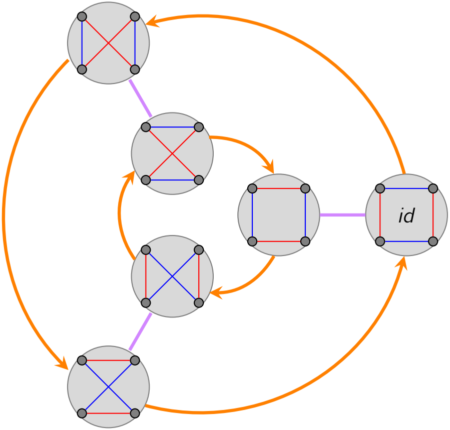
The holomorph of V_4

We've also seen the first step of the construction of

$$\mathrm{Hol}(V_4) = V_4 \rtimes \mathrm{Aut}(V_4) \cong V_4 \rtimes D_3.$$

$\mathrm{Aut}(V_4) \cong D_3$

	id	α	α^2	β	$\alpha\beta$	$\alpha^2\beta$
id	id	α	α^2	β	$\alpha\beta$	$\alpha^2\beta$
α	α	α^2	id	$\alpha\beta$	$\alpha^2\beta$	β
α^2	α^2	id	α	$\alpha^2\beta$	β	$\alpha\beta$
β	β	$\alpha^2\beta$	$\alpha\beta$	id	α^2	α
$\alpha\beta$	$\alpha\beta$	β	$\alpha^2\beta$	α	id	α^2
$\alpha^2\beta$	$\alpha^2\beta$	$\alpha\beta$	β	α^2	α	id



Replacing each orange and purple edge with four will complete the contruction.

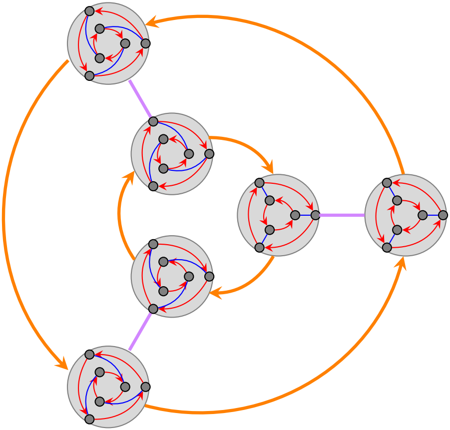
The holomorph of D_3

We've also seen the first step of the construction of

$$\mathrm{Hol}(D_3) = D_3 \rtimes \mathrm{Aut}(D_3) \cong D_3 \rtimes D_3.$$

$\mathrm{Aut}(D_3) \cong D_3$

	id	α	α^2	β	$\alpha\beta$	$\alpha^2\beta$
id	id	α	α^2	β	$\alpha\beta$	$\alpha^2\beta$
α	α	α^2	id	$\alpha\beta$	$\alpha^2\beta$	β
α^2	α^2	id	α	$\alpha^2\beta$	β	$\alpha\beta$
β	β	$\alpha^2\beta$	$\alpha\beta$	id	α^2	α
$\alpha\beta$	$\alpha\beta$	β	$\alpha^2\beta$	α	id	α^2
$\alpha^2\beta$	$\alpha^2\beta$	$\alpha\beta$	β	α^2	α	id



Replacing each orange and purple edge with six will complete the contruction.

Internal semidirect products

Remark

In the semidirect product $A \rtimes_\theta B$, the subgroups $A \times \{1\}$ and $\{1\} \times B$

- generate $A \rtimes_\theta B$:
$$\big(A \times \{1\}\big)\big(\{1\} \times B\big) = A \rtimes_\theta B$$
- intersect trivially:
$$\big(A \times \{1\}\big) \cap \big(\{1\} \times B\big) = \{(1,1)\}.$$
- one is normal: $A \times \{1\}$.

In the next lecture, we'll see that if $G = NH$, with

- $G = NH$
- $N \cap H = \{1\}$
- $N \trianglelefteq G$,

then $G \cong N \rtimes_\theta H$, where $\theta \colon H \to \mathrm{\mathbf{Inn}}(N) \leq \mathrm{\mathbf{Aut}}(N)$.

This is called an inner or internal semidirect product.

This condidtion can easily be checked in the subgroup lattice, by inspection.

It will also imply that if $Z(G) = \{e\}$, then $\mathrm{\mathbf{Hol}}(G) \cong G \times G$, and so

$$\mathrm{\mathbf{Hol}}(D_3) \cong D_3 \rtimes D_3 \cong D_3 \times D_3.$$