Visual Algebra

Lecture 4.10: Internal products

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Motivation and overview

We've seen how to define the direct product $A \times B$ of two arbitrary groups.

This is called an external (or outer) direct product.

Sometimes, a group is secretly the direct product of two subgroups: $G = NH \cong N \times H$.

This is called an internal (or inner) direct product.

We've seen how to define an external semidirect product $A \rtimes_\theta B$ of two arbitrary groups.

We'll also learn when G is an internal semidirect product of subgroups: $G = NH \cong N \rtimes H$.

The labeling map $H \longrightarrow \mathrm{Aut}(N)$ sends h to an inner automorphism.

Inner direct and semidirect products can be identified by inspection of the subgroup lattice.

We'll also learn about **central products**, both external and internal.

Internal products

Previously, we've looked at outer products: taking two unrelated groups and constructing a direct or semidirect product.

Now, we'll explore when a group $G = NH$ is isomorphic to a direct or semidirect product.

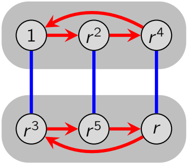
These are called internal products. Let's see two examples:

$\theta_0 \colon r \mapsto \varphi^0$

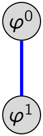
$\mathbf{G = C_6}$

$\mathbf{N = \langle r^2 \rangle}$

$\mathbf{H = \langle r^3 \rangle}$



$C_6 = NH \cong N \times H$

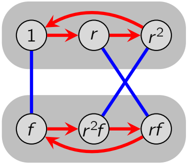


$\theta_1 \colon r \mapsto \varphi$

$\mathbf{G = D_3}$

$\mathbf{N = \langle r \rangle}$

$\mathbf{H = \langle f \rangle}$



$D_3 = NH \cong N \rtimes_\theta H$

Questions

- Can we characterize when $NH \cong N \times H$ and/or $NH \cong N \rtimes_\theta H$?
- If $NH \cong N \rtimes_\theta H$, then what is the map $\theta \colon H \to \mathrm{Aut}(N)$?

Internal direct products

When $G = NH$ is isomorphic to $N \times H$, we have an isomorphism

$$i \colon N \times H \longrightarrow NH, \qquad i \colon (n,h) \longmapsto nh.$$

Since $N \times \{1\}$ and $\{1\} \times H$ are normal in $N \times H$, the subgroups N and H are normal in NH.

Recall that earlier, we showed that

$$|NH| = \frac{|N| \cdot |H|}{|N \cap H|},$$

and so it follows that if $NH \cong N \times H$, then $N \cap H = \{e\}$.

Theorem

Let $N, H \leq G$. Then $G \cong N \times H$ iff the following conditions hold:

(i) N and H are normal (ii) $N \cap H = \{e\}$ (iii) $G = NH$.

Remark

This has a very nice interpretation in terms of subgroup lattices! Subgroups for which (*ii*) and (*iii*) hold are called lattice complements.

Internal semidirect products

When $G = NH$ is isomorphic to $N \rtimes_\theta H$, we have an isomorphism

$$i \colon N \rtimes_\theta H \longrightarrow NH, \qquad i \colon (n,h) \longmapsto nh.$$

This time, only $N \times \{1\}$ needs to be normal in $N \rtimes_\theta H$, and so $N \trianglelefteq NH$.

As before, from

$$|NH| = \frac{|N| \cdot |H|}{|N \cap H|},$$

we conclude that if $NH \cong N \rtimes_\theta H$, then $N \cap H = \{e\}$.

Theorem

Let $N, H \leq G$. Then $G \cong N \rtimes H$ iff the following conditions hold:

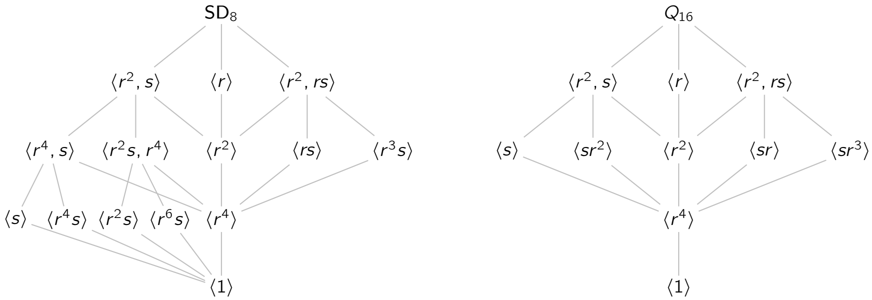
(i) N is normal in G (ii) $N \cap H = \{e\}$ (iii) $G = NH$,

and the homomorphism θ sends h to the inner automorphism $\varphi_{h^{-1}}$:

$$\theta \colon H \longrightarrow \mathrm{Aut}(N), \qquad \theta \colon h \longmapsto \big(n \overset{\varphi_{h^{-1}}}{\longmapsto} hnh^{-1}\big).$$

Let's do several examples for intution, before proving this.

Examples of internal semidirect products



Observations

- The group SD_8 decomposes as a semidirect product several ways:

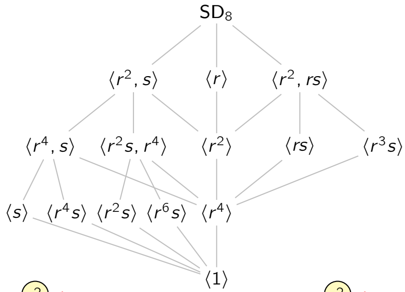
$$N = \langle r \rangle \cong C_8, \quad H = \langle s \rangle \cong C_2, \qquad \mathrm{SD}_8 = NH \cong C_8 \rtimes_{\theta_3} C_2.$$

  or alternatively,

$$N = \langle r^2, rs \rangle \cong Q_8, \quad H = \langle s \rangle \cong C_2, \qquad \mathrm{SD}_8 = NH \cong Q_8 \rtimes_{\theta'} C_2.$$

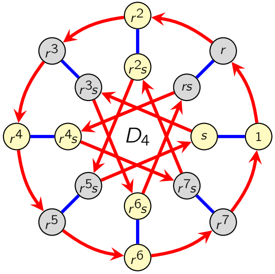
- The group Q_{16} does *not* decompose as a semidirect product!

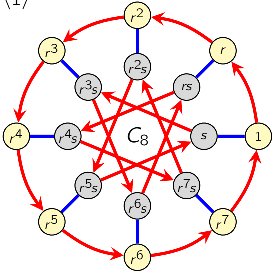
Semidihedral groups as semidirect products

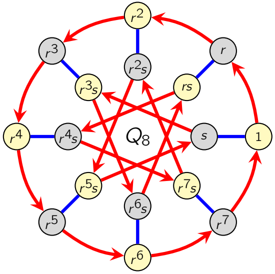


$$SD_8 \cong \langle r\rangle \rtimes \langle s\rangle \cong C_8 \rtimes C_2$$

$$SD_8 \cong \langle r^2, rs\rangle \rtimes \langle s\rangle \cong Q_8 \rtimes C_2$$



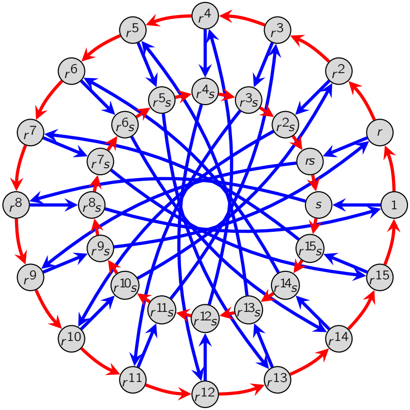


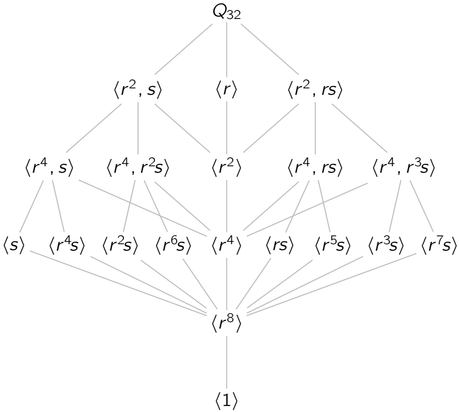


Generalized quaternion groups

Recall that a generalized quaternion group is a dicyclic group whose order is a power of 2.

It's not hard to see that $r^8 = s^2 = -1$ is contained in every cyclic subgroup.





Therefore, $Q_{2^n} \ncong N \rtimes H$ for any of its nontrivial subgroups.

Lattice complements, both normal

Lemma

Let $H, N \leq G$ be lattice complements. These are normal iff $hn = nh$ for all $h \in H$, $n \in N$.

Proof

"$\Rightarrow$:" Since $H, N \trianglelefteq G$,

$$[n, h] = nhn^{-1}h^{-1} = n\underbrace{(hn^{-1}h^{-1})}_{\in N} = \underbrace{(nhn^{-1})}_{\in H}h^{-1} \in H \cap N = \{e\}. \qquad \checkmark$$

"$\Leftarrow$:" Suppose each $[n, h] = e$.

For an arbitrary $g = nh \in G$,

$$nhH = nH = \big\{nh \mid h \in H\big\} = \big\{hn \mid h \in H\big\} = Hn \qquad \Longrightarrow \qquad H \trianglelefteq G.$$

By symmetry, N must be normal. ✓

Lattice complements, both normal

Theorem

Let $N, H \leq G$. Then $G \cong N \times H$ iff the following conditions hold:

(i) N, H are normal (ii) $N \cap H = \{e\}$ (iii) $G = NH$.

Proof

Since N is normal, $G = NH$. Define the map

$$i\colon N \times H \longrightarrow NH, \qquad i\colon (n,h) \longmapsto nh,$$

Homomorphism: Since elements in N and H pairwise commute,

$$i\big((n_1,h_1)\cdot(n_2,h_2)\big) = i\big((n_1n_2, h_1h_2)\big) = n_1n_2h_1h_2 = n_1h_1n_2h_2 = i\big((n_1,h_1)\big)\cdot i\big((n_2,h_2)\big). \ \checkmark$$

Onto: $nh \in NH$ has preimage $(n,h) \in N \times H$. ✓

1-to-1: Suppose $i\big((n_1,h_1)\big) = i\big((n_2,h_2)\big)$, or equivalently, $n_1h_1 = n_2h_2$.

Then $n_2^{-1}n_1 = h_2h_1^{-1} \in N \cap H = \{e\}$, so $n_1 = n_2$ and $h_1 = h_2$. ✓

Lattice complements, one normal

Theorem

Let $N, H \leq G$. Then $G \cong N \rtimes H$ iff the following conditions hold:

(i) N is normal in G (ii) $N \cap H = \{e\}$ (iii) $G = NH$,

and the homomorphism θ sends h to the inner automorphism $\varphi_{h^{-1}}$:

$$\theta\colon H \longrightarrow \mathrm{Aut}(N), \qquad \theta\colon h \longmapsto \big(n \stackrel{\varphi_{h^{-1}}}{\longmapsto} hnh^{-1}\big).$$

Proof

Define the map

$$i\colon N \rtimes_\theta H \longrightarrow NH, \qquad i\colon (n,h) \longmapsto nh,$$

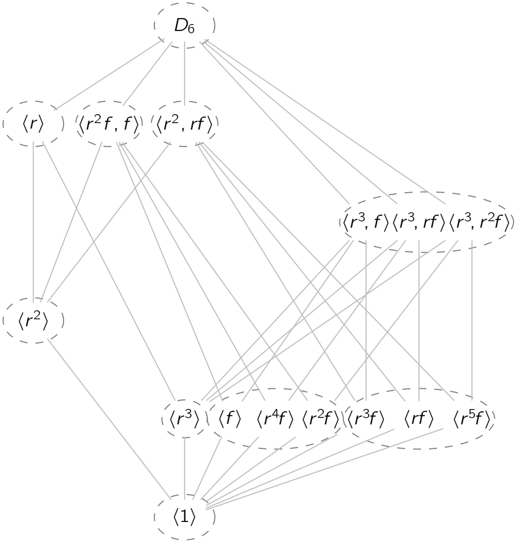
Homomorphism: $i\big((n_1,h_1)\big) \cdot i\big((n_2,h_2)\big) = n_1h_1n_2h_2$, and

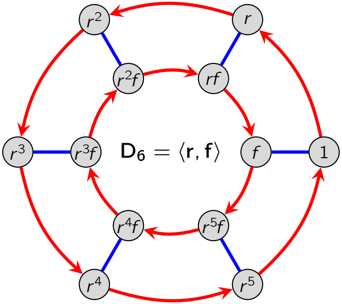
$$i\big((n_1,h_1)*(n_2,h_2)\big) = i\big((n_1\underbrace{h_1n_2h_1^{-1}}_{=\varphi_{h_1^{-1}}(n_2)}, h_1h_2)\big) = n_1h_1n_2h_1^{-1}h_1h_2 = n_1h_1n_2h_2.$$

Bijective: Analogous to the direct product case. □

Internal direct and semidirect products

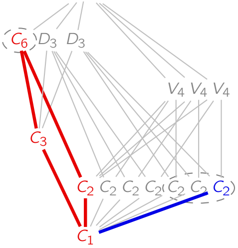
In how many ways does D_6 decompose as a direct or semidirect product of its subgroups?



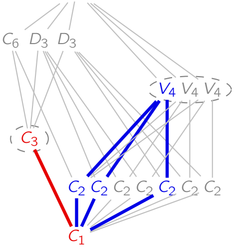


Decompositions of D_6 into direct and semdirect products

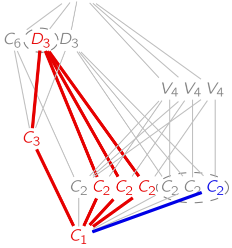
$D_6 \cong C_6 \rtimes C_2$



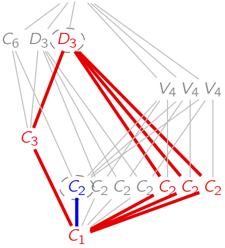
$D_6 \cong C_3 \rtimes V_4$



$D_6 \cong D_3 \rtimes C_2$

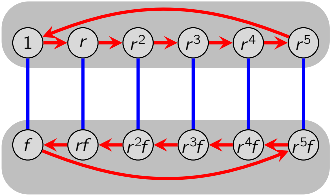


$D_6 \cong D_3 \times C_2$

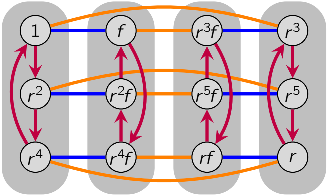


Decompositions of D_6 into direct and semdirect products

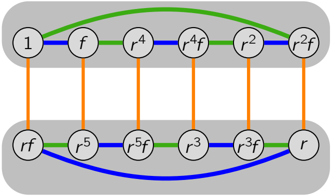
$C_6 \rtimes C_2$



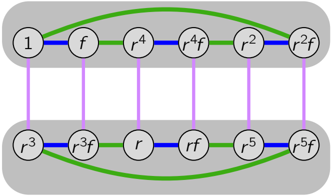
$C_3 \rtimes V_4$



$D_3 \rtimes C_2$



$D_3 \times C_2$



Central products

The following 3 conditions characterize when $G = NH \cong N \times H$.

1. H and N are normal,
2. $G = \langle H, N \rangle$,
3. $H \cap N = \langle 1 \rangle$.

If we weaken the first to only N being normal, we get $G = NH \cong N \rtimes H$.

Alernatively, we can keep the first two but weaken the third.

Definition

Suppose H and N are subgroups of G satisfying:

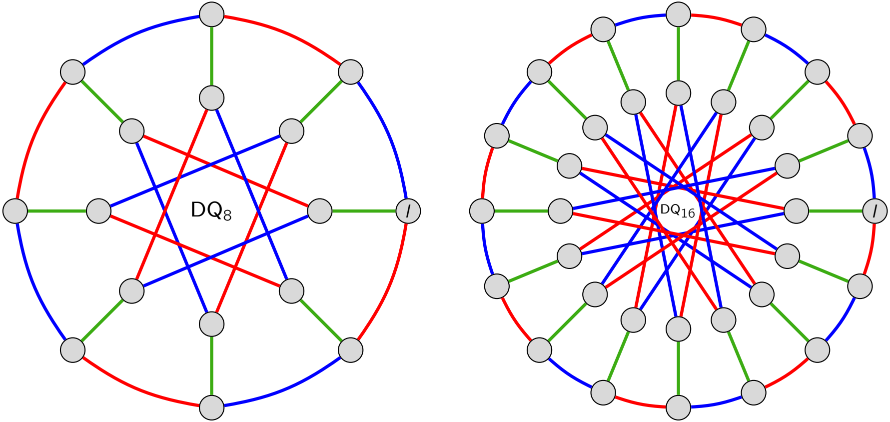
1. H and N are normal,
2. $G = \langle H, N \rangle$,
3. $H \cap N \leq Z(G)$.

The G is an internal central product of N and H, denoted $G \cong N \circ H$.

We can also define an *external central product* of A and B, but we won't do that here.

Revisiting the diquaternion groups

How many semidirect products can you find of the form $H \rtimes_\theta C_2$, just by inspection?



Do you see any central products?

Central products

The diquaternion group DQ_8 is a central product two nontrivial ways:

- $\mathrm{DQ}_8 \cong C_4 \circ D_4$
- $\mathrm{DQ}_8 \cong C_4 \circ Q_8$.

Recall that $Z(\mathrm{DQ}_8) = N \cong C_4$.