Visual Algebra

Lecture 4.11: Homomorphisms in surprising locations

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Motivation and overview

Sometimes, when learning abstract algebra, it can feel a little detached from other areas of mathematics.

The concept of a homomorphism arises in a number of surprising places.

We'll see it appear in the following areas, presented in reverse chronological order as people typically learn them:

- Linear algebra
- Differential equations
- Integral calculus
- Logarithms
- Trig identities

Homomorphisms in linear algebra

Consider a system of equations, represented in matrix form by $\mathbf{Ax} = \mathbf{b}$.

The matrix $\mathbf{A}$ represents a linear map $\mathbb{R}^n \to \mathbb{R}^m$. This is a homomorphism, because

$$\mathbf{A}(\mathbf{u} + \mathbf{v}) = \mathbf{Au} + \mathbf{Av}, \qquad \text{for all } \mathbf{u}, \mathbf{v} \in \mathbb{R}^n.$$

The kernel is often called the nullspace, $\mathbf{x}_n$, which is the general solution to $\mathbf{Ax} = \mathbf{0}$.

The general solution $\mathbf{Ax} = \mathbf{b}$ is the preimage of $\mathbf{b}$.

This is the coset of $\text{Ker}(\mathbf{A})$:

$$\mathbf{x} = \mathbf{x}_p + \text{Ker}(\mathbf{A}) = \{\mathbf{x}_p + \mathbf{z} \mid \mathbf{Az} = \mathbf{0}\},$$

where $\mathbf{x}_p(t)$ is any particular solution to the original ODE.

Homomorphisms in differential equations

Consider the differential equation $x'' + 4x = 12$.

Let $D = \frac{d^2}{dt^2} + 4$, a differential operator. This is a homomorphism

$$D \colon \mathcal{C}^2(\mathbb{R}) \longrightarrow \mathcal{C}^0(\mathbb{R}), \qquad D \colon f(t) \longmapsto f''(t) + 4f(t).$$

The kernel is

$$\text{Ker}(D) = \big\{ f(t) \mid f''(t) + 4f(t) = 0 \big\} = \big\{ A\cos(2t) + B\sin(2t) \mid A, B \in \mathbb{R} \big\}.$$

The general solution to the original ODE $Dx = 12$ is the preimage of 12.

This set contains $x(t) = 3$. The general solution is the coset of $\text{Ker}(D)$ containing this:

$$3 + \text{Ker}(D) = \big\{ A\cos(2t) + B\sin(2t) + 3 \mid A, B \in \mathbb{R} \big\}.$$

More generally, the general solution of a linear ODE $Dx = h(t)$ is

$$x(t) = \text{Ker}(D) + x_p(t) = x_h(t) + x_p(t),$$

where $x_p(t)$ is any particular solution to the original ODE, and $x_h(t)$ solves $Dx = 0$.

Homomorphisms in calculus

Let $\mathcal{C}^1(\mathbb{R})$ be the group of differentiable real-valued functions.

Let $\mathcal{C}^0(\mathbb{R})$ be the group of continuous real-valued functions.

Consider the differential operator

$$\frac{d}{dx}\colon \mathcal{C}^1(\mathbb{R}) \longrightarrow \mathcal{C}^0(\mathbb{R}), \qquad \frac{d}{dx}\colon f(x) \longmapsto f'(x).$$

This a homomorphism because

$$\frac{d}{dx}\big(f+g\big) = \frac{df}{dx} + \frac{dg}{dx}.$$

The kernel is the subgroup C of constant functions

$$\text{Ker}(\tfrac{d}{dx}) = \{c \in \mathbb{R}\}.$$

The preimage of $2x \in \mathcal{C}^0(\mathbb{R})$, denoted $\int 2x\,dx$, is a coset of the kernel that contains x^2:

$$\int 2x\,dx \; = \; x^2 + \text{Ker}\left(\tfrac{d}{dx}\right) = x^2 + C.$$

Homomorphisms involving exponentials and logarithms

Consider the expoential function e^x as a map between groups

$$\exp\colon (\mathbb{R},+) \longrightarrow (\mathbb{R}^*,\times), \qquad \exp\colon x \longmapsto e^x.$$

This is a homomorphism because

$$\exp(x+y) = e^{x+y} = e^x e^y = \exp(x)\exp(y).$$

Since it is a bijection, there is an inverse map, that we'll call ln:

$$\ln\colon (\mathbb{R}^*,\times) \longrightarrow (\mathbb{R},+).$$

Since this is a homomorphism, it must satisfy

$$\ln(xy) = \ln x + \ln y.$$

Homomorphisms and trig identities

Consider the function

$$R\colon (\mathbb{R},+) \longrightarrow \mathsf{Mat}_2(\mathbb{R}), \qquad R\colon \theta \longmapsto \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

This is a homomorphism, because rotating by α, and then by β:

$$\underbrace{\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}}_{\text{rotate by } \alpha}\underbrace{\begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}}_{\text{rotate by } \beta} = \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -(\cos\alpha\sin\beta + \sin\alpha\cos\beta) \\ \cos\alpha\sin\beta + \sin\alpha\cos\beta & \cos\alpha\cos\beta - \sin\alpha\sin\beta \end{bmatrix}$$

is the same as rotating by $\alpha+\beta$

$$R(\alpha+\beta) = \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix} = R(\alpha)R(\beta).$$

Equating entries gives the trig identies for $\cos(\alpha+\beta)$ and $\sin(\alpha+\beta)$.