Visual Algebra

Lecture 5.2: Five features of group actions

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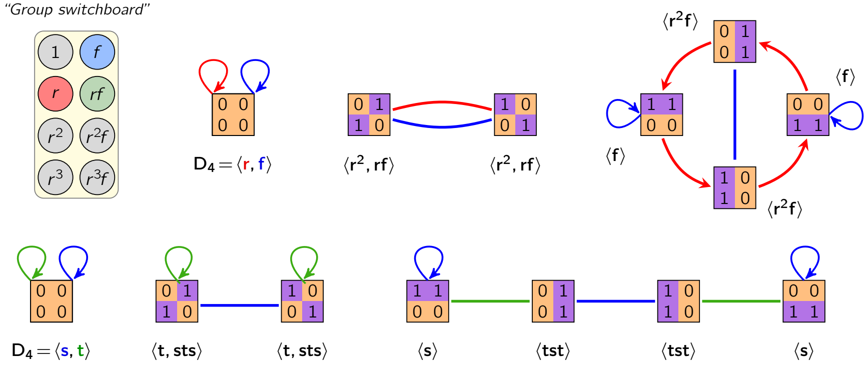
Group actions, action graphs, and G-sets

Definition

A set S with an action by G is called a (right) G-set.

Big ideas

- An action $\phi\colon G \to \mathrm{Perm}(S)$ endows S with an **algebraic structure**.
- *Action graphs are to G-sets, like how Cayley graphs are to groups.*



Five features of every group action

Every group action has **five fundamental features** that we will always try to understand.

There are several ways to classify them. For example:

- three are subsets of S
- two are subgroups of G.

Another way to distinguish them is by local vs. global:

- three are features of individual group or set elements (we'll write in *lowercase*)
- two are features of the homomorphism ϕ. (we'll write in *Uppercase*)

We will see parallels within and between these classes.

For example, two "local" features will be "dual" to each other, as will the global features.

Our global features can be expressed as intersections of our local features, either ranging over all $s \in S$, or over all $g \in G$.

We'll start by exploring the three local features.

Notation

Throughout, we'll denote identity elements by $1 \in G$ and $e \in \mathrm{Perm}(S)$.

Two local features: orbits and stabilizers

Suppose G acts on a set S, and pick some $s \in S$. We can ask two questions about it:

(i) What other states (in S) are reachable from s? (We call this the orbit of s.)

(ii) What group elements (in G) fix s? (We call this the stabilizer of s.)

Definition

Suppose that G acts on a set S (on the right) via $\phi\colon G \to \mathrm{Perm}(S)$.

(i) The orbit of $s \in S$ is the set

$$\mathrm{orb}(s) = \{s.\phi(g) \mid g \in G\}.$$

(ii) The stabilizer of s in G is

$$\mathrm{stab}(s) = \{g \in G \mid s.\phi(g) = s\}.$$

In terms of the action graph

(i) The orbit of $s \in S$ is the connected component containing s.

(ii) The stabilizer of $s \in S$ are the group elements whose paths start and end at s; "loops."

The third local feature: fixators

Our first two local features were specific to a certain set element $s \in S$.

Our last local feature is defined for each group element $g \in G$. A natural question to ask is:

(iii) What *states* (in S) does g fix?

Definition

Suppose that G acts on a set S (on the right) via $\phi \colon G \to \mathrm{Perm}(S)$.

(iii) The fixator of $g \in G$ are the elements $s \in S$ fixed by g:

$$\mathrm{fix}(g) = \{s \in S \mid s.\phi(g) = s\}.$$

In terms of the action graph

(iii) The fixator of $g \in G$ are the nodes from which the g-paths are loops.

In terms of the "group switchboard analogy"

(i) The orbit of $s \in S$ are the elements in S that can be reached by pressing some combination of buttons.

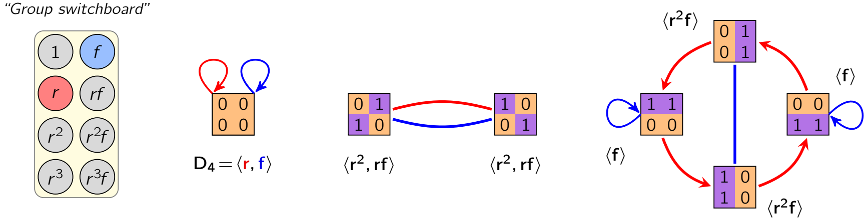
(ii) The stabilizer of $s \in S$ consists of the buttons that have no effect on s.

(iii) The fixator of $g \in G$ are the elements in S that don't move when we press the g-button.

Three local features: orbits, stabilizers, and fixators

The orbits of our running example are the 3 connected components.

Each node is labeled by its stabilizer.



The fixators are $\text{fix}(1) = S$, and

$$\text{fix}(r) = \text{fix}(r^3) = \left\{ \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \right\} \qquad \text{fix}(r^2) = \text{fix}(rf) = \text{fix}(r^3 f) = \left\{ \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}, \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}, \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right\}$$

$$\text{fix}(f) = \left\{ \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}, \begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}, \begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix} \right\} \qquad \text{fix}(r^2 f) = \left\{ \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}, \begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}, \begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix} \right\}$$

Local duality: stabilizers vs. fixators

Consider the following table, where a checkmark at (g, s) means "*g fixes s*."

	$\begin{smallmatrix}0&0\\0&0\end{smallmatrix}$	$\begin{smallmatrix}0&1\\1&0\end{smallmatrix}$	$\begin{smallmatrix}1&0\\0&1\end{smallmatrix}$	$\begin{smallmatrix}0&0\\1&1\end{smallmatrix}$	$\begin{smallmatrix}0&1\\0&1\end{smallmatrix}$	$\begin{smallmatrix}1&1\\0&0\end{smallmatrix}$	$\begin{smallmatrix}1&0\\1&0\end{smallmatrix}$
1	✓	✓	✓	✓	✓	✓	✓
r	✓						
r^2	✓	✓	✓				
r^3	✓						
f	✓			✓		✓	
rf	✓	✓	✓				
r^2f	✓				✓		✓
r^3f	✓	✓	✓				

- the stablizers can be read off the columns: *group elements that fix $s \in S$*
- the fixators can be read off the rows: *set elements fixed by $g \in G$.*

The stabilizer subgroup

Notice how in our example, the stabilizer of each $s \in S$ is a subgroup.

This holds true for any action.

Proposition

For any $s \in S$, the set **stab**(s) is a subgroup of G.

Proof (outline)

To show **stab**(s) is a group, we need to show three things:

(i) **Identity**. That is, $s.\phi(1) = s$.

(ii) **Inverses**. That is, if $s.\phi(g) = s$, then $s.\phi(g^{-1}) = s$.

(iii) **Closure**. That is, if $s.\phi(g) = s$ and $s.\phi(h) = s$, then $s.\phi(gh) = s$.

Alternatively, it suffices to show that if $s.\phi(g) = s$ and $s.\phi(h) = s$, then $s.\phi(gh^{-1}) = s$,

All three of these are very intuitive in our our switchboard analogy.

The stabilizer subgroup

As we've seen, elements in the same orbit can have different stabilizers.

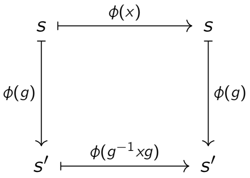
Proposition (exercise)

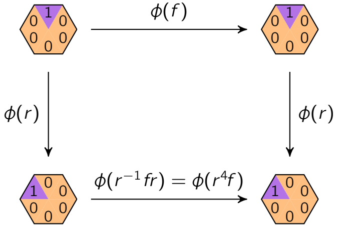
Set elements in the same orbit have conjugate stabilizers:

$$\text{stab}(s.\phi(g)) = g^{-1}\,\text{stab}(s)g, \quad \text{for all } g \in G \text{ and } s \in S.$$

In other words, if x stabilizes s, then $g^{-1}xg$ stabilizes $s.\phi(g)$.

Here are several ways to visualize what this means and why.



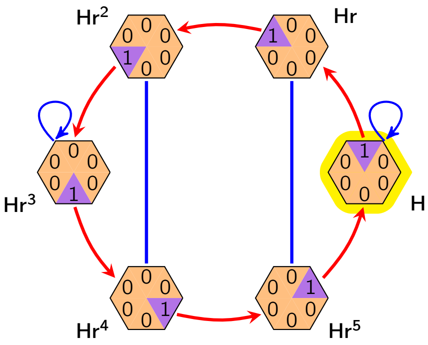


In other words, if x is a loop from s, and $s \xrightarrow{g} s'$, then $g^{-1}xg$ is a loop from s'.

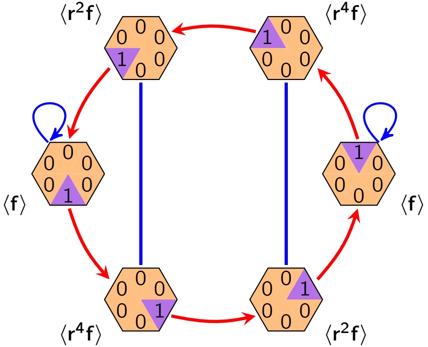
The stabilizer subgroup

Here is another example of an action (or G-set), this time of $G = D_6$.

Let s be the highlighted hexagon, and $H = \mathbf{stab}(s)$.



labeled by destinations



labeled by stabilizers

Two global features: fixed points and the kernel

Our last two features are properties of the action ϕ, rather than of specific elements.

One definition is new, and the other is an familiar concept in this new setting.

Definition

Suppose that G acts on a set S via $\phi\colon G \to \mathsf{Perm}(S)$.

(iv) The kernel of the action is the set

$$\mathsf{Ker}(\phi) = \big\{k \in G \mid \phi(k) = e\big\} = \big\{k \in G \mid s.\phi(k) = s \text{ for all } s \in S\big\}.$$

(v) The fixed points of the action, denoted $\mathsf{Fix}(\phi)$, are the orbits of size 1:

$$\mathsf{Fix}(\phi) = \big\{s \in S \mid s.\phi(g) = s \text{ for all } g \in G\big\}.$$

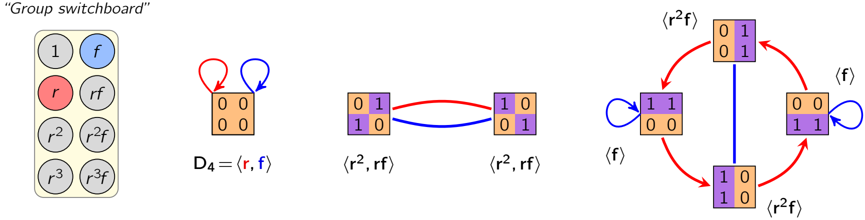
Proposition (global duality: fixed points vs. kernel)

Suppose that G acts on a set S via $\phi\colon G \to \mathsf{Perm}(S)$. Then

$$\mathsf{Ker}(\phi) = \bigcap_{s \in S} \mathsf{stab}(s), \qquad \text{and} \qquad \mathsf{Fix}(\phi) = \bigcap_{g \in G} \mathsf{fix}(g).$$

Let's also write $\mathsf{Orb}(\phi)$ for the set of orbits of ϕ.

Two global features: fixed points and the kernel



In terms of the action graph

(iv) The kernel of ϕ are the paths that are "loops from every $s \in S$."

(v) The fixed points of ϕ are the size-1 connected components.

In terms of the group switchboard analogy

(iv) The kernel of ϕ are the "broken buttons"; those $g \in G$ that have no effect on any s.

(v) The fixed points of ϕ are those $s \in S$ that are not moved by pressing any button.

Global duality: fixed points vs. kernel

Consider the following table, where a checkmark at (g, s) means g fixes s.

	0 0 / 0 0	0 1 / 1 0	1 0 / 0 1	0 0 / 1 1	0 1 / 0 1	1 1 / 0 0	1 0 / 1 0
1	✓	✓	✓	✓	✓	✓	✓
r	✓						
r^2	✓	✓	✓				
r^3	✓						
f	✓			✓		✓	
rf	✓	✓	✓				
r^2f	✓				✓		✓
r^3f	✓	✓	✓				

- the fixed points consist of columns with all checkmarks: *set elts fixed by everything*
- the kernel consists of the rows with all checkmarks: *group elements that fix everything*.