Visual Algebra

Lecture 5.3: Two theorems on orbits

Dr. Matthew Macauley

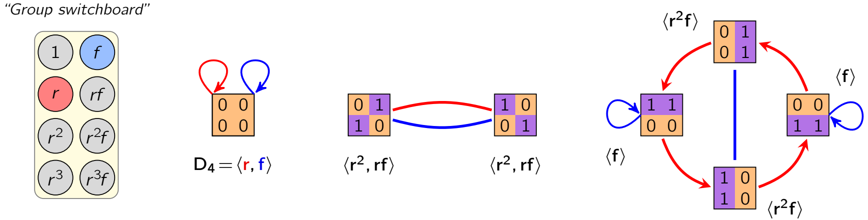
School of Mathematical & Statistical Sciences
Clemson University
South Carolina, USA
http://www.math.clemson.edu/~macaule/

Motivation

Our binary square example gives us some key intutition about group actions.

Qualitative Observation 1

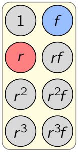
Elements in larger orbits tend to have smaller stabilizers, and vice-versa

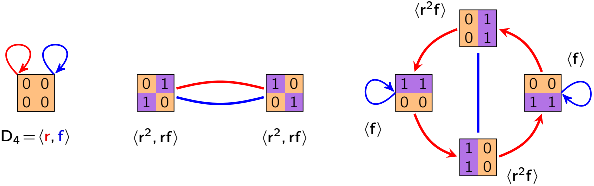


Qualitative Observation 2

Actions whose fixed point tables have more "checkmarks" tend to have more orbits.

"Group switchboard"





	0 0 0 0	0 1 1 0	1 0 0 1	0 0 1 1	0 1 0 1	1 1 0 0	1 0 1 0
1	✓	✓	✓	✓	✓	✓	✓
r	✓						
r^2	✓	✓	✓				
r^3	✓						
f	✓			✓		✓	
rf	✓	✓	✓				
r^2f	✓				✓		✓
r^3f	✓	✓	✓				

Two theorems on orbits, and their consequences

Qualitative observations

- elements in larger orbits tend to have smaller stabilizers, and vice-versa
- actions whose fixed point tables have more "checkmarks" tend to have more orbits.

Both of these qualitative observations can be formalized into quantitative theorems.

Theorems

1. **Orbit-stabilizer theorem**: the size of an orbit is the index of the stabilizer.
2. **Orbit-counting theorem**: the number of orbits is the average number of things fixed by a group element.

If we set up our group actions correctly, the orbit-stabilizer theorem will imply:

- The size of the conjugacy class $\mathrm{cl}_G(H)$ is the index of the normalizer of $H \leq G$
- The size of the conjugacy class $\mathrm{cl}_G(x)$ is the index of the centralizer of $x \in G$

We can also determine the number of conjugacy classes from the orbit-counting theorem.

Our first theorem on orbits

Orbit-stabilizer theorem

For any group action $\phi\colon G \to \mathrm{Perm}(S)$, and $s \in S$, the size of the orbit containing s is

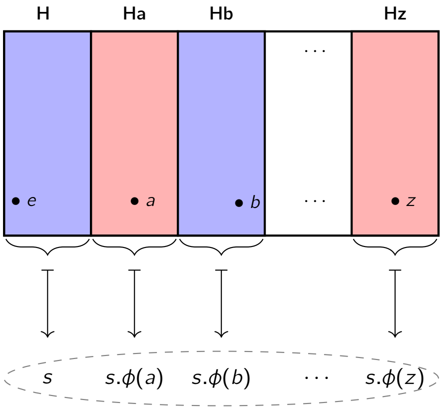
$$|\operatorname{orb}(s)| = [G : \operatorname{stab}(s)].$$

By Lagrange's theorem, this says that $|\operatorname{orb}(s)| \cdot |\operatorname{stab}(s)| = |G|$.

Let $\mathbf{H} = \operatorname{stab}(\mathbf{s})$

applying to $s \in S$ anything in this coset of $\operatorname{stab}(s)$ . . .

. . . yields this element in $\operatorname{orb}(s)$



$[G : \operatorname{stab}(s)]$ *cosets*

$\big|\operatorname{orb}(s)\big|$ *elements*

Our first theorem on orbits

Orbit-stabilizer theorem

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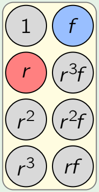
By Lagrange's theorem, this says that $|\,\text{orb}(s)| \cdot |\,\text{stab}(s)| = |G|$.

Proof

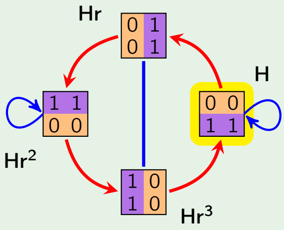
Goal: Exhibit a bijection between elements of $\text{orb}(s)$, and right cosets of $\text{stab}(s)$.

That is, "*two g-buttons send s to the same place iff they're in the same coset*".

"Group switchboard"



1	f	$\mathbf{H = stab(s)}$
r	fr	$\mathbf{Hr}$
r^2	fr^2	$\mathbf{Hr^2}$
r^3	fr^3	$\mathbf{Hr^3}$



Note that $s.\phi(a) = s.\phi(b)$ iff a and b are in the same right coset of H in G.

The orbit-stabilizer theorem: $|\text{orb}(s)| = [G : \text{stab}(s)]$

Let $H\backslash G$ denote the set of right cosets of H in G. [Recall: G/H is the set of left cosets.]

Proof

Throughout, let $H = \text{stab}(s)$. Define a map

$$f \colon H\backslash G \longrightarrow \text{orb}(s), \qquad f \colon Hg \longmapsto s.\phi(g).$$

Well-defined: Suppose $Ha = Hb$. Then

$$\begin{aligned}
Hab^{-1} = H \quad &\Longrightarrow \quad ab^{-1} \in H && \text{(by the "boring but useful coset lemma")} \\
&\Longrightarrow \quad s.\phi(ab^{-1}) = s && \text{(by definition of stabilzer)} \\
&\Longrightarrow \quad s.\phi(a)\phi(b^{-1}) = s && \text{(properties of homomorphisms)} \\
&\Longrightarrow \quad s.\phi(a)\phi(b)^{-1} = s && \text{(properties of homomorphisms)} \\
&\Longrightarrow \quad s.\phi(a) = s.\phi(b) && \text{(right-multiply by } \phi(b)) \\
&\Longrightarrow \quad f(Ha) = f(Hb) && \text{(by definition of } f)
\end{aligned}$$

One-to-one: Change each $\Longrightarrow$ into $\Longleftrightarrow$. ✓

Onto: The preimage of $s' = s.\phi(g)$ is Hg. ✓

If we have instead, a left group action, the proof carries through but using left cosets.

Our second theorem on orbits

Orbit-counting theorem

Let a finite group G act on a set S via $\phi\colon G \to \mathsf{Perm}(S)$. Then

$$|\operatorname{Orb}(\phi)| = \frac{1}{|G|}\sum_{g\in G} |\operatorname{fix}(g)|.$$

This says that the "*average number of checkmarks per row*" is the number of orbits:

	$\begin{smallmatrix}0&0\\0&0\end{smallmatrix}$	$\begin{smallmatrix}0&1\\1&0\end{smallmatrix}$	$\begin{smallmatrix}1&0\\0&1\end{smallmatrix}$	$\begin{smallmatrix}0&0\\1&1\end{smallmatrix}$	$\begin{smallmatrix}0&1\\0&1\end{smallmatrix}$	$\begin{smallmatrix}1&1\\0&0\end{smallmatrix}$	$\begin{smallmatrix}1&0\\1&0\end{smallmatrix}$
1	✓	✓	✓	✓	✓	✓	✓
r	✓						
r^2	✓	✓	✓				
r^3	✓						
f	✓			✓		✓	
rf	✓	✓	✓				
r^2f	✓				✓		✓
r^3f	✓	✓	✓				

Orbit-counting theorem: $|\text{Orb}(\phi)| = \frac{1}{|G|}\sum_{g\in G} |\text{fix}(g)|$.

Proof

Let's first count the number of checkmarks in the fixed point table, three ways:

$$\underbrace{\sum_{g\in G} |\text{fix}(g)|}_{\text{count by rows}} = \Big|\{(g,s)\in G\times S \mid s.\phi(g)=s\}\Big| = \underbrace{\sum_{s\in S} |\text{stab}(s)|}_{\text{count by columns}} .$$

By the orbit-stabilizer theorem, we can replace each $|\text{stab}(s)|$ with $|G|/|\text{orb}(s)|$:

$$\sum_{s\in S} |\text{stab}(s)| = \sum_{s\in S}\frac{|G|}{|\text{orb}(s)|} = |G|\sum_{s\in S}\frac{1}{|\text{orb}(s)|}.$$

Let's express this sum over all disjoint orbits $S=\mathcal{O}_1\cup\cdots\cup\mathcal{O}_k$ separately:

$$|G|\sum_{s\in S}\frac{1}{|\text{orb}(s)|} = |G|\sum_{\mathcal{O}\in\text{Orb}(\phi)}\Big(\underbrace{\sum_{s\in\mathcal{O}}\frac{1}{|\text{orb}(s)|}}_{=1\quad(\textit{why?})}\Big) = |G|\sum_{\mathcal{O}\in\text{Orb}(\phi)} 1 = |G|\cdot|\text{Orb}(\phi)|.$$

Equating this last term with the first term gives the desired result. □