Visual Algebra

Lecture 5.4: Examples of group actions

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Groups acting on elements, subgroups, and cosets

It is frequently of interest to analyze the action of a group G on its elements, subgroups, or cosets of some fixed $H \leq G$.

Often, the orbits, stabilizers, and fixed points of these actions are familiar algebraic objects.

A number of deep theorems have a slick proof via a clever group action.

Here are common examples of group actions:

- G acts on itself by multiplication.
- G acts on itself by conjugation.
- G acts on its subgroups by conjugation.
- G acts on the cosets of a fixed subgroup $H \leq G$ by multiplication.

For each of these, we'll characterize the orbits, stabilizers, fixators, fixed points, and kernel.

We'll encounter familiar objects such as conjugacy classes, normalizers, stabilziers, and normal subgroups, as some of our "five fundamental features".

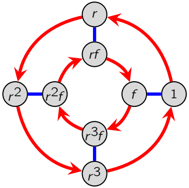
Theorems that we have observed but haven't been able to prove yet will fall in our lap!

Groups acting on themselves by multiplication

Assume $|G| > 1$. The group G acts on itself (that is, $S = G$) by **right-multiplication**:

$$\phi\colon G \longrightarrow \text{Perm}(S)\,, \qquad \phi(g) = \text{the permutation that sends each } x \mapsto xg.$$

- there is only one orbit: $\text{orb}(x) = G$, for all $x \in G$
- the stabilizer of each $x \in G$ is $\text{stab}(x) = \langle 1 \rangle$
- the fixator of $g \neq 1$ is $\text{fix}(g) = \emptyset$.
- there are no fixed points, and the kernel is trivial:



$$\text{Fix}(\phi) = \bigcap_{g \in G} \text{fix}(g) = \emptyset, \qquad \text{and} \qquad \text{Ker}(\phi) = \bigcap_{s \in S} \text{stab}(s) = \langle 1 \rangle.$$

Cayley's theorem

If $|G| = n$, then there is an embedding $G \hookrightarrow S_n$.

Proof

Let G act on itself by right multiplication. This defines a homomorphism

$$\phi\colon G \longrightarrow \text{Perm}(S) \cong S_n.$$

Since $\text{Ker}(\phi) = \langle 1 \rangle$, it is an embedding. □

Groups acting on themselves by conjugation

Another way a group G can act on itself (that is, $S = G$) is by **right-conjugation**:

$$\phi \colon G \longrightarrow \mathrm{Perm}(S)\,, \qquad \phi(g) = \text{the permutation that sends each } x \mapsto g^{-1}xg.$$

- The orbit of $x \in G$ is its conjugacy class:

$$\mathrm{orb}(x) = \big\{x.\phi(g) \mid g \in G\big\} = \big\{g^{-1}xg \mid g \in G\big\} = \mathrm{cl}_G(x).$$

- The stabilizer of x is its centralizer:

$$\mathrm{stab}(x) = \big\{g \in G \mid g^{-1}xg = x\big\} = \big\{g \in G \mid xg = gx\big\} := C_G(x)$$

- The fixator of $g \in G$ is also its centralizer, because

$$\mathrm{fix}(g) = \big\{x \in S \mid x.\phi(g) = x\big\} = \big\{x \in G \mid g^{-1}xg = x\big\} = C_G(g).$$

- The fixed points and kernel are the center, because

$$\mathrm{Fix}(\phi) = \bigcap_{g \in G} \mathrm{fix}(g) = \bigcap_{g \in G} C_G(g) = Z(G) = \bigcap_{x \in G} C_G(x) = \bigcap_{x \in G} \mathrm{stab}(x) = \mathrm{Ker}(\phi).$$

Groups acting on themselves by conjugation

Let's apply our two theorems:

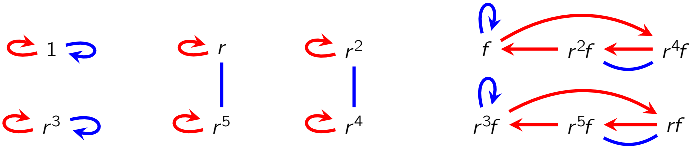
1. **Orbit-stabilizer theorem**. "*the size of an orbit is the index of the stabilizer*":

$$\left|\,\mathrm{cl}_G(x)\right| = [G : C_G(x)] = \frac{|G|}{|C_G(x)|}.$$

2. **Orbit-counting theorem**. "*the number of orbits is the average number of elements fixed by a group element*":

$$\#\text{conjugacy classes of } G = \text{average size of a centralizer.}$$

Let's revisit our old example of conjugacy classes in $D_6 = \langle r, f \rangle$:



Notice that the stabilizers are $\mathrm{stab}(r) = \mathrm{stab}(r^2) = \mathrm{stab}(r^4) = \mathrm{stab}(r^5) = \langle r \rangle$,

$$\mathrm{stab}(1) = \mathrm{stab}(r^3) = D_6, \qquad \mathrm{stab}(r^i f) = \langle r^3, r^i f \rangle.$$

Groups acting on themselves by conjugation

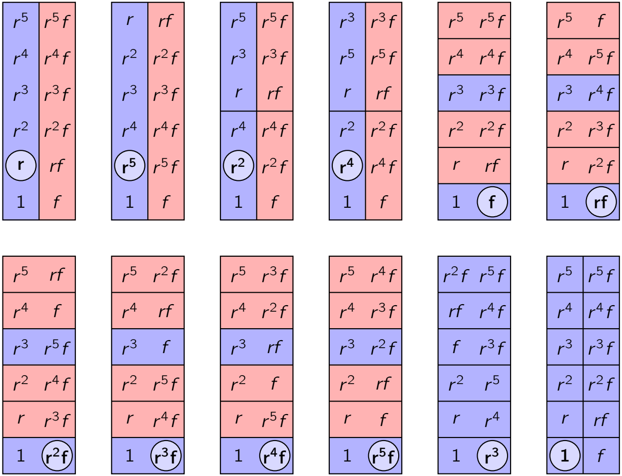
Here is the "*fixed point table*". Note that $\mathrm{Ker}(\phi) = \mathrm{Fix}(\phi) = \langle r^3 \rangle$.

	1	r	r^2	r^3	r^4	r^5	f	rf	r^2f	r^3f	r^4f	r^5f
1	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
r	✓	✓	✓	✓	✓	✓						
r^2	✓	✓	✓	✓	✓	✓						
r^3	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
r^4	✓	✓	✓	✓	✓	✓						
r^5	✓	✓	✓	✓	✓	✓						
f	✓			✓			✓			✓		
rf	✓			✓				✓			✓	
r^2f	✓			✓					✓			✓
r^3f	✓			✓			✓			✓		
r^4f	✓			✓				✓			✓	
r^5f	✓			✓					✓			✓

By the **orbit-counting theorem**, there are $|\,\mathrm{Orb}(\phi)| = 72/|D_6| = 6$ conjugacy classes.

Groups acting on themselves by conjugation

Here are the cosets of all 12 cyclic subgroups in D_6 (some coincide).



Do you see how to deduce from the orbit-counting theorem that there are 6 conjugacy classes?

Groups acting on subgroups by conjugation

Any group G acts on its set S of subgroups by **right-conjugation**:

$$\phi\colon G \longrightarrow \mathrm{Perm}(S)\,, \qquad \phi(g) = \text{the permutation that sends each } H \text{ to } g^{-1}Hg.$$

This is a **right action**, but there is an associated left action: $H \mapsto gHg^{-1}$.

Let $H \leq G$ be an element of S.

- The orbit of H consists of all conjugate subgroups:
$$\mathrm{orb}(H) = \left\{g^{-1}Hg \mid g \in G\right\} = \mathrm{cl}_G(H).$$
- The stabilizer of H is the normalizer of H in G:
$$\mathrm{stab}(H) = \left\{g \in G \mid g^{-1}Hg = H\right\} = N_G(H).$$
- The fixator of g are the subgroups that g normalizes:
$$\mathrm{fix}(g) = \left\{H \mid g^{-1}Hg = H\right\} = \left\{H \mid g \in N_G(H)\right\}.$$
- The fixed points of ϕ are precisely the normal subgroups of G:
$$\mathrm{Fix}(\phi) = \left\{H \leq G \mid g^{-1}Hg = H \text{ for all } g \in G\right\}.$$
- The kernel of this action is the set of elements that normalize every subgroup:
$$\mathrm{Ker}(\phi) = \left\{g \in G \mid g^{-1}Hg = H \text{ for all } H \leq G\right\} = \bigcap_{H \leq G} N_G(H).$$

Groups acting on subgroups by conjugation

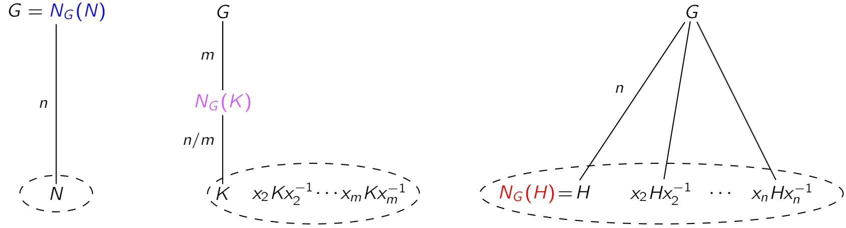
Let's apply our two theorems:

1. **Orbit-stabilizer theorem**. "*the size of an orbit is the index of the stabilizer*":

$$\big|\,\mathrm{cl}_G(H)\big| = [G : N_G(H)] = \frac{|G|}{|N_G(H)|}.$$

2. **Orbit-counting theorem**. "*the number of orbits is the average number of elements fixed by a group element*":

$$\#\text{conjugacy classes of subgroups of } G = \mathbb{E}\big[\#\text{ subgroups } g \text{ normalizes}\big].$$



normal $\big|\,\mathrm{cl}_G(N)\big| = 1$

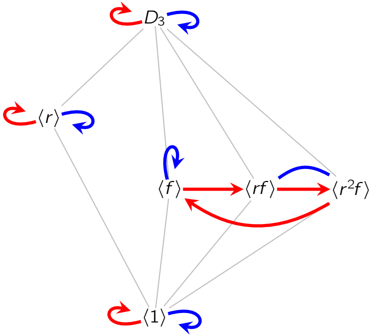
moderately unnormal $1 < \big|\,\mathrm{cl}_G(K)\big| < [G : K]$

fully unnormal $\big|\,\mathrm{cl}_G(H)\big| = [G : H]$; as large as possible

Groups acting on subgroups by conjugation

Here is an example of $G = D_3$ acting on its subgroups.

$\tau(1)$	$=$	$\langle 1\rangle$	$\langle r\rangle$	$\langle f\rangle$	$\langle rf\rangle$	$\langle r^2f\rangle$	D_3
$\tau(r)$	$=$	$\langle 1\rangle$	$\langle r\rangle$	$\langle f\rangle$	$\langle rf\rangle$	$\langle r^2f\rangle$	D_3
$\tau(r^2)$	$=$	$\langle 1\rangle$	$\langle r\rangle$	$\langle f\rangle$	$\langle rf\rangle$	$\langle r^2f\rangle$	D_3
$\tau(f)$	$=$	$\langle 1\rangle$	$\langle r\rangle$	$\langle f\rangle$	$\langle rf\rangle$	$\langle r^2f\rangle$	D_3
$\tau(rf)$	$=$	$\langle 1\rangle$	$\langle r\rangle$	$\langle f\rangle$	$\langle rf\rangle$	$\langle r^2f\rangle$	D_3
$\tau(r^2f)$	$=$	$\langle 1\rangle$	$\langle r\rangle$	$\langle f\rangle$	$\langle rf\rangle$	$\langle r^2f\rangle$	D_3



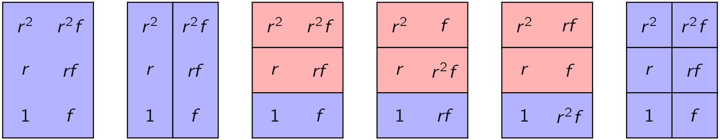
Observations

Do you see how to read stabilizers and fixed points off of the permutation diagram?

- $\mathrm{Ker}(\phi) = \langle 1\rangle$ consists of the row(s) with only fixed points.
- $\mathrm{Fix}(\phi) = \{\langle 1\rangle, \langle r\rangle, D_3\}$ consists of the column(s) with only fixed points.
- By the orbit-counting theorem, there are $|\mathrm{Orb}(\phi)| = 24/|D_3| = 4$ conjugacy classes.

Groups acting on subgroups by conjugation

Consider the partitions of D_3 by the left cosets of its six subgroups:



D_3/D_3 $\quad$ $D_3/\langle r\rangle$ $\quad$ $D_3/\langle f\rangle$ $\quad$ $D_3/\langle rf\rangle$ $\quad$ $D_3/\langle r^2f\rangle$ $\quad$ $D_3/\langle 1\rangle$

- $\text{fix}(g)$ are the subgroups H for which "*g appears in a blue coset of H*"
- $\text{Ker}(\phi)$ are elements that "*only appear in blue cosets*"
- By the orbit-counting theorem, the subgroups fall into

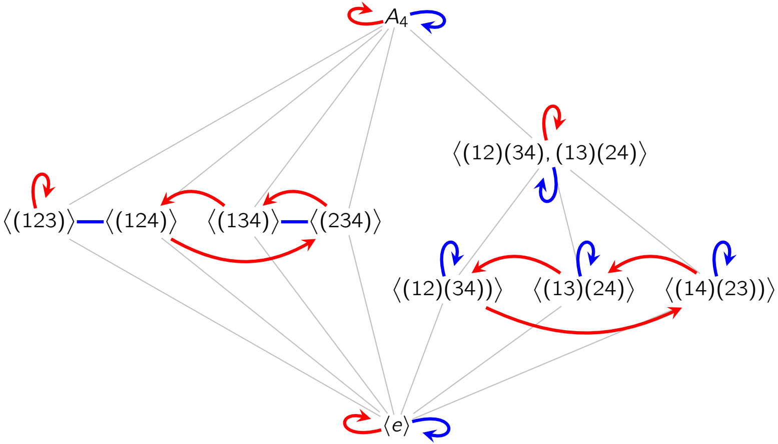
$$|\,\text{Orb}(\phi)| = \text{average \# checkmarks per row} = \frac{\text{total \# of blue entries}}{|G|}$$

conjugacy classes.

Equivalently: *how many full "G-boxes" the blue cosets can be rearranged to fill up.*

Groups acting on subgroups by conjugation

Here is an example of $G = A_4 = \langle (123), (12)(34) \rangle$ acting on its subgroups.



Let's take a moment to revisit our "*three favorite examples*" from Chapter 3.

$$N = \langle (12)(34), (13)(24) \rangle, \qquad H = \langle (123) \rangle, \qquad K = \langle (12)(34) \rangle.$$

Groups acting on subgroups by conjugation

Here is the "*fixed point table*" of the action of A_4 on its subgroups.

	$\langle e \rangle$	$\langle (123) \rangle$	$\langle (124) \rangle$	$\langle (134) \rangle$	$\langle (234) \rangle$	$\langle (12)(34) \rangle$	$\langle (13)(24) \rangle$	$\langle (14)(23) \rangle$	$\langle (12)(34), (13)(24) \rangle$	A_4
e	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
(123)	✓	✓							✓	✓
(132)	✓	✓							✓	✓
(124)	✓		✓						✓	✓
(142)	✓		✓						✓	✓
(134)	✓			✓					✓	✓
(143)	✓			✓					✓	✓
(234)	✓				✓				✓	✓
(243)	✓				✓				✓	✓
$(12)(34)$	✓					✓	✓	✓	✓	✓
$(13)(24)$	✓					✓	✓	✓	✓	✓
$(14)(23)$	✓					✓	✓	✓	✓	✓

By the **orbit-counting theorem**, there are $|\operatorname{Orb}(\phi)| = 60/|A_4| = 5$ conjugacy classes.

Groups acting on cosets of H by multiplication

Fix a subgroup $H \leq G$. Then G acts on its **right cosets** by **right-multiplication**:

$$\phi\colon G \longrightarrow \mathrm{Perm}(S), \qquad \phi(g) = \text{the permutation that sends each } Hx \text{ to } Hxg.$$

Let Hx be an element of $S = H\backslash G$ (the right cosets of H).

- There is only one orbit. For example, given two cosets Hx and Hy,

$$\phi(x^{-1}y) \;\text{ sends }\; Hx \longmapsto Hx(x^{-1}y) = Hy.$$

- The stabilizer of Hx is the conjugate subgroup $x^{-1}Hx$:

$$\mathrm{stab}(Hx) = \{g \in G \mid Hxg = Hx\} = \{g \in G \mid Hxgx^{-1} = H\} = x^{-1}Hx.$$

- There doesn't seem to be a standard term for the fixator of g:

$$\mathrm{fix}(g) = \{Hx \mid Hxg = Hx\} = \{Hx \mid xgx^{-1} \in H\}.$$

- Assuming $H \neq G$, there are no fixed points of ϕ.
- The kernel of this action is the intersection of all conjugate subgroups of H:

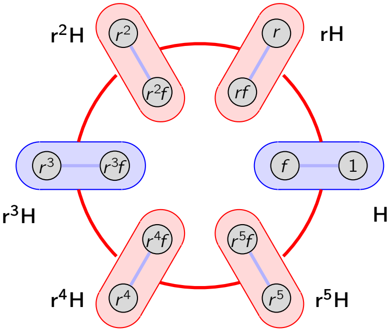
$$\mathrm{Ker}(\phi) = \bigcap_{x \in G} \mathrm{stab}(x) = \bigcap_{x \in G} x^{-1}Hx.$$

Notice that $\langle 1 \rangle \leq \mathrm{Ker}\,\phi \leq H$, and $\mathrm{Ker}(\phi) = H$ iff $H \trianglelefteq G$.

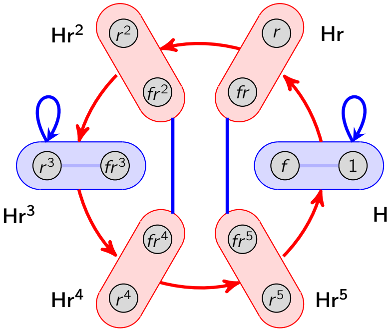
Groups acting on cosets of H by multiplication

The quotient process is done by collapsing the Cayley graph by the left cosets of H.

In contrast, this action is the result of collapsing the Cayley graph by the right cosets.



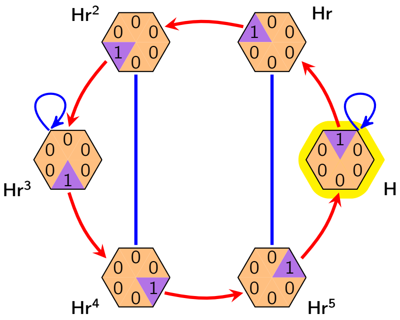
not a valid action graph

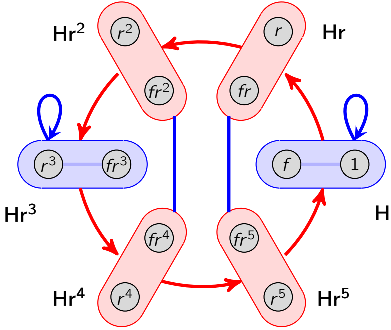


action graph of ϕ

Groups acting on cosets of H by multiplication

Soon, we'll see that *every* transitive action is equivalent to G acting on cosets of a subgroup.





This is why it's helpful to have a notion of G-set isomorphism.

In other words, we can *always* quotient by a subgroup $H \leq G$ to get a G-set.

This G-set is a group if and only if H is normal.

A summary of our four actions

We have seen four important (right) actions of a group G, acting on:

- itself by multiplication
- itself by conjugation
- its subgroups by conjugation
- cosets of $H \leq G$ by multiplication.

set $S =$	G		subgroups of G	right cosets of H
operation	multiplication	conjugation	conjugation	right multiplication
$\text{orb}(s)$	G	$\text{cl}_G(g)$	$\text{cl}_G(H)$	$H\backslash G$
$\lvert\text{orb}(s)\rvert$	$\lvert G\rvert$	$[G : C_G(g)]$	$[G : N_G(H)]$	$[G : H]$
$\lvert\text{Orb}(\phi)\rvert$	1	avg. $\lvert\text{cl}_G(g)\rvert$	avg. $\lvert\text{cl}_G(H)\rvert$	1
$\text{stab}(s)$	$\langle 1\rangle$	$C_G(g)$	$N_G(H)$	$x^{-1}Hx$
$\text{fix}(g)$	$\{1\}$ or $\emptyset$	$C_G(g)$	$\{H \mid g \in N_G(H)\}$	$\{Hx \mid xgx^{-1} \in H\}$
$\text{Fix}(\phi)$	none	$Z(G)$	normal subgroups	none
$\text{Ker}(\phi)$	$\langle 1\rangle$	$Z(G)$	$\displaystyle\bigcap_{H\leq G} N_G(H)$	$\displaystyle\bigcap_{x\leq G} x^{-1}Hx$