Visual Algebra

Lecture 5.5: Actions of automorphisms

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Actions of automorphism groups

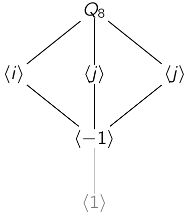
For any G, the automorphism group $\mathrm{Aut}(G)$ naturally acts on $S = G$ via a homomorphism

$$\phi\colon \mathrm{Aut}(G) \longrightarrow \mathrm{Perm}(S), \qquad \phi(\sigma) = \textit{the permutation that sends each } g \mapsto \sigma(g).$$

Let's see an example. Any $\sigma \in \mathrm{Aut}(Q_8)$ must send i to an element of order 4: $\pm i$, $\pm j$, $\pm k$.

This leaves 4 choices for $\sigma(j)$. Therefore, $|\,\mathrm{Aut}(Q_8)| \leq 24$.

The inner automorphism group is $N := \mathrm{Inn}(Q_8) = \{\, \mathrm{Id}, \varphi_i, \varphi_j, \varphi_k \,\}$.



$\mathrm{Inn}(Q_8) \cong Q_8/\langle -1 \rangle \cong V_4$

Z	iZ	jZ	kZ
1	i	j	k
-1	$-i$	$-j$	$-k$

cosets of $Z(Q_8)$ are in bijection with inner automorphisms of Q_8

cl (1)	1	i	j	k
cl (−1)	-1	$-i$	$-j$	$-k$
		cl (i)	**cl (j)**	**cl (k)**

inner automorphisms of Q_8 permute elements within conjugacy classes

All 6 permutations of $\{i, j, k\}$ define a subgroup $H \leq \mathrm{Aut}(Q_8)$. Since $N \cap H = \langle \mathrm{Id} \rangle$,

$$\mathrm{Aut}(Q_8) \cong \mathrm{Inn}(Q_8) \rtimes \underbrace{H}_{\cong S_3} = \mathrm{Inn}(Q_8) \rtimes \mathrm{Out}(Q_8) \cong V_4 \rtimes S_3 \cong S_4.$$

Automorphisms of Q_8

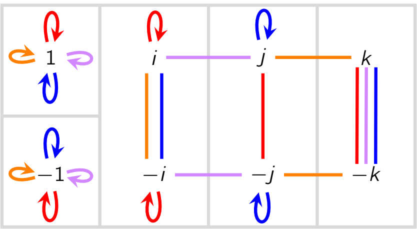
The group $\mathrm{Aut}(Q_8)$ naturally acts on the set S of...

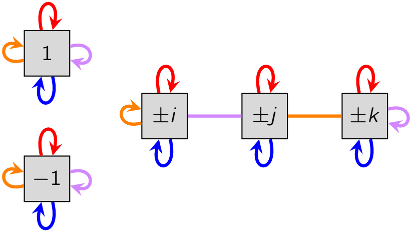
- elements of Q_8, via

$$\phi\colon \mathrm{Aut}(G) \longrightarrow \mathrm{Perm}(S), \qquad \phi(\sigma) = \textit{the permutation that sends each } g \mapsto \sigma(g).$$

- conjugacy classes of Q_8, via

$$\theta\colon \mathrm{Aut}(G) \longrightarrow \mathrm{Perm}(S), \qquad \theta(\sigma) = \textit{the permutation sending each } \mathrm{cl}_G(g) \mapsto \mathrm{cl}_G(\sigma(g)).$$

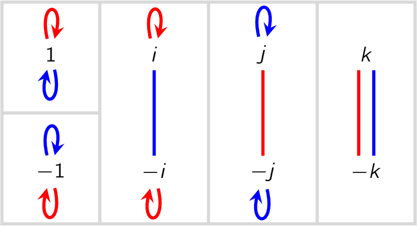




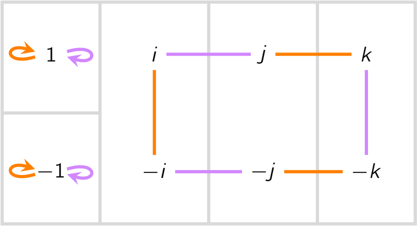
Automorphisms of Q_8

There are also actions by the inner and outer automorphism groups.

$\mathbf{Inn}(Q_8) \cong V_4$ *acting on* $S = Q_8$.



$\mathbf{Out}(Q_8) \cong S_3$ *does not act on* $S = Q_8$



These groups can also act on the:

- conjugacy classes of G,
- set of subgroups of G.

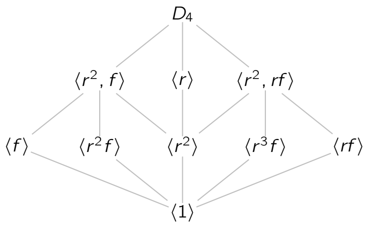
Characteristic subgroups

Definition

A subgroup $H \leq G$ is characteristic, written $H \,\text{char}\, G$ or $H \blacktriangleleft G$, if $\sigma(H) = H$ for all $\sigma \in \text{Aut}(G)$.

Examples of characteristic subgroups are the center $Z(G)$ and commutator subgroup G'.

Normality is *not* transitive: $K \trianglelefteq H \trianglelefteq G$ does not imply $K \trianglelefteq G$.

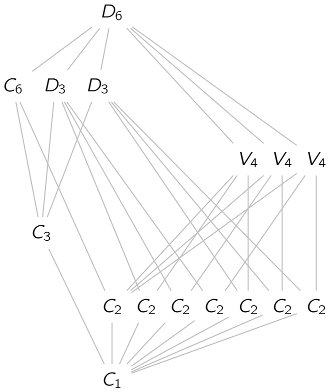


Proposition

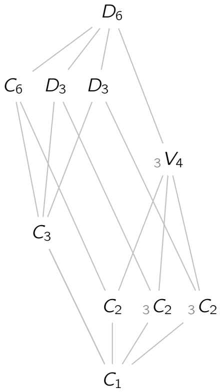
Being characteristic *is* transitive: $K \blacktriangleleft H \blacktriangleleft G$ implies $K \blacktriangleleft G$.

Characterstic subgroup diagrams

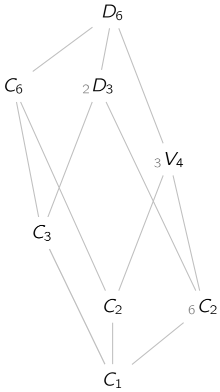
Sometimes, it is helpful to see a subgroup diagram variant, where the nodes are automorphs, instead of conjugacy classes.



subgroup lattice



subgroup diagram



automorph diagram

Other characteristic subgroups

A maximal subgroup of G is some $M \leq G$ for which $M \leq H \leq G$ implies $H = M$ or $H = G$.

Definition

The Frattini subgroup, denoted $\Phi(G)$, is the intersection of all maximal subgroups of G.

Properties

- $\Phi(G)$ is characteristic, and hence normal.
- $\Phi(G)$ is the set of non-generating elements of G:

$$\Phi(G) = \big\{a \in G \mid \text{if } a \in S \text{ and } G = \langle S \rangle, \text{ then } G = \langle S \backslash \{a\} \rangle\big\}.$$

- If H and K are finite, then $\Phi(H \times K) = \Phi(H) \times \Phi(K)$.

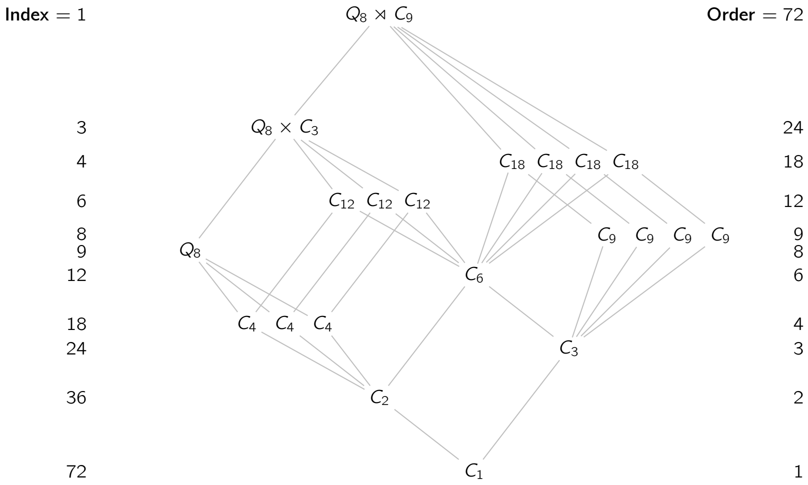
Definition

The scole, denoted $\text{soc}(G)$, is the generated by all minimal normal subgroups of G.

If G is a finite solvable group, then $\text{soc}(G)$ is a product of cyclic groups of prime order.

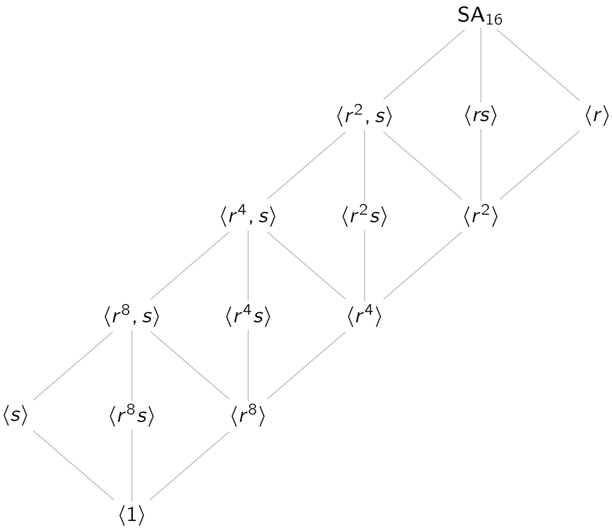
Examples

Let's compute the center, commutator subgroup, Frattini subgroup, socle of $G = Q_8 \rtimes C_9$.



Examples

Let's compute the center, commutator subgroup, Frattini subgroup, socle of $G = \mathrm{SA}_{16}$.



Examples

Let's compute the center, commutator subgroup, Frattini subgroup, socle of $G = C_7 \rtimes C_6$.