Visual Algebra

Lecture 5.6: Action equivalence and G-set isomorphism

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Action equivalence

Let's recall the difference between left-conjugating and right conjugating:

"Left group action"

$$\begin{tikzcd} H \arrow[r, "\text{conjugate by } a"] \arrow[rr, bend right, "\text{conjugate by } ba"'] & aHa^{-1} \arrow[r, "\text{conjugate by } b"] & baHa^{-1}b^{-1} \end{tikzcd}$$

$$\phi(a)\phi(b) = \phi(ba)$$

"Right group action"

$$\begin{tikzcd} H \arrow[r, "\text{conjugate by } a"] \arrow[rr, bend right, "\text{conjugate by } ab"'] & a^{-1}Ha \arrow[r, "\text{conjugate by } b"] & b^{-1}a^{-1}Hab \end{tikzcd}$$

$$\phi(a)\phi(b) = \phi(ab)$$

There's a better way to describe left actions than the faux-homomorphic $\phi(a)\phi(b) = \phi(ba)$.

"Left group action"

$$\begin{tikzcd} H \arrow[r, "\text{right-conjugate by } a^{-1}"] \arrow[rr, bend right, "\text{right-conj. by } (a^{-1}b^{-1})^{-1}"'] & aHa^{-1} \arrow[r, "\text{right-conjugate by } b^{-1}"] & baHa^{-1}b^{-1} \end{tikzcd}$$

$$\phi(a^{-1})\phi(b^{-1}) = \phi(a^{-1}b^{-1}) = \phi((ba)^{-1})$$

"Right group action"

$$\begin{tikzcd} H \arrow[r, "\text{right-conjugate by } a"] \arrow[rr, bend right, "\text{right-conj. by } ab"'] & a^{-1}Ha \arrow[r, "\text{right-conjugate by } b"] & b^{-1}a^{-1}Hab \end{tikzcd}$$

$$\phi(a)\phi(b) = \phi(ab)$$

Big idea

For every right action, there is an "equivalent" left-action where:

"pressing g-buttons, from L-to-R" $\Leftrightarrow$ "pressing g^{-1}-buttons, from R-to-L".

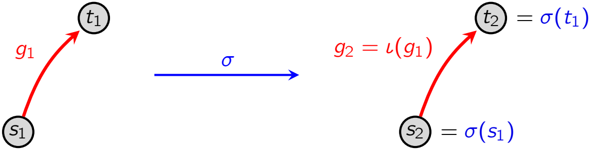
Action equivalence, informally

Action equivalence is more general. Consider two groups acting on sets, say via

$$\phi_1 \colon G_1 \longrightarrow \mathbf{Perm}(S_1), \qquad \text{and} \qquad \phi_2 \colon G_2 \longrightarrow \mathbf{Perm}(S_2).$$

If these are "equivalent", then we'll need

- a set bijection $\sigma \colon S_1 \longrightarrow S_2$
- a group isomorphism $\iota \colon G_1 \longrightarrow G_2$.



Informally, these actions are **equivalent** if:

1. pressing the g_1-button in the G_1-switchboard, followed by
2. applying $\sigma \colon S_1 \to S_2$ to get to the other graph

is the same as doing these steps in reverse order. That is,

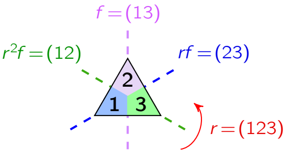
1. applying $\sigma \colon S_1 \to S_2$ to get to the other graph, then
2. pressing the $\iota(g_1)$-button on the G_2-switchboard.

A familiar example of equivalent actions

We've seen the groups:

- D_3 act on a set X of six triangles,
- S_3 act on a set X' of six permuations of **123**.

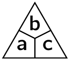
These two actions are equivalent.



$$X' = \{\mathbf{123}, \mathbf{132}, \mathbf{213}, \mathbf{231}, \mathbf{312}, \mathbf{321}\}$$

Set bijection

$\sigma :$



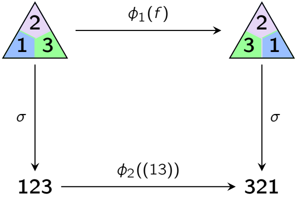
$\longmapsto$ **abc**

Isomorphism

$\iota \colon D_3 \longrightarrow S_3$

$\iota \colon r \longmapsto (123)$

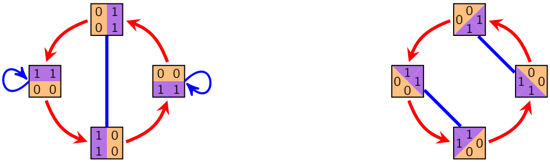
$\iota \colon f \longmapsto (23)$



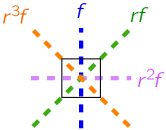
Equivalence of actions

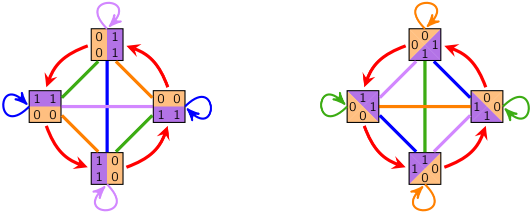
Consider the following two sets:

Should the following two D_4-actions be considered "equivalent"?



What if we add generators?





Action equivalence, formally

Definition

Two actions $\phi_1 \colon G_1 \longrightarrow \textbf{Perm}(S_1)$ and $\phi_2 \colon G_2 \longrightarrow \textbf{Perm}(S_2)$ are equivalent if there is an isomorphism $\iota \colon G_1 \to G_2$ and a bijection $\sigma \colon S_1 \to S_2$ such that

$$\sigma \circ \phi_1(g) = \phi_2(\iota(g)) \circ \sigma, \quad \text{for all } g \in G.$$

We say that the resulting action graphs are action equivalent.

If $G_1 = G_2$ and $\iota \colon G \to G$ is the identity map, then S_1 and S_2 are isomorphic as G-sets.

This can be expressed with a commutative diagram:

$$\begin{tikzcd} S_1 \arrow[r, "\phi_1(g)"] \arrow[d, "\sigma"'] & S_1 \arrow[d, "\sigma"] \\ S_2 \arrow[r, "\phi_2(\iota(g))"'] & S_2 \end{tikzcd}$$

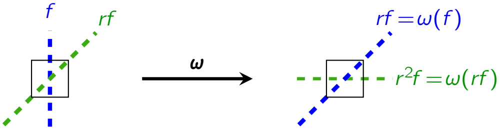
Action equivalence can be used to show that in our binary square example, we could have:

- defined $\phi(r)$ to rotate clockwise, and $\phi(f)$ to flip vertically
- used tiles with a and b, rather than 0 and 1
- read from right-to-left, rather than left-to-right, etc.

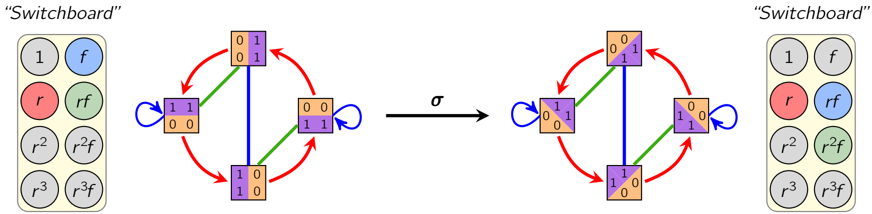
Equivalence of actions

Consider the following two sets:

The map $\sigma\colon S \longrightarrow S'$ and outer automorphism $\omega \in \mathrm{Aut}(D_4)$



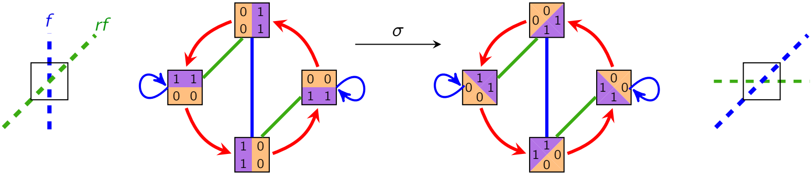
define an equivalence between the following actions:



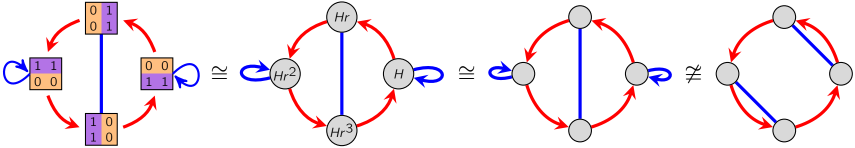
Action equivalence (weaker) vs. G-set isomorphism (stronger)

Just like we did for groups, we formalized what it means for G-sets to be isomorphic.

Since $\iota\colon G \to G$ must be the identity, the following is *not* a G-set isomorphism:



Therefore, the following equivalent actions, are as D_4-sets:



Every right action has an equivalent left action

G acting on...	right action	equivalent left action
itself by multiplication	$x \mapsto xg$	$x \mapsto g^{-1}x$
itself by conjugation	$x \mapsto g^{-1}xg$	$x \mapsto gxg^{-1}$
its subgroups by conjugation	$H \mapsto g^{-1}Hg$	$H \mapsto gHg^{-1}$
cosets by multiplication	$Hx \mapsto Hxg$	$xH \mapsto g^{-1}xH$

$$\begin{tikzcd}
x \arrow[r, mapsto, "\phi_R(g)"] \arrow[d, mapsto, "\sigma"'] & xg \arrow[d, mapsto, dashed, "\text{not } \sigma"] \\
x^{-1} \arrow[r, mapsto, "\theta(g)"] & gx^{-1}
\end{tikzcd}$$

$$\begin{tikzcd}
x \arrow[r, mapsto, "\phi_R(g)"] \arrow[d, mapsto, "\sigma"'] & xg \arrow[d, mapsto, "\sigma"] \\
x^{-1} \arrow[r, mapsto, "\phi_L(g)"] & g^{-1}x^{-1}
\end{tikzcd}$$

$$\begin{tikzcd}
x \arrow[r, mapsto, "\phi_R(g)"] \arrow[d, mapsto, "\text{Id}"'] & xg \arrow[d, mapsto, dashed, "\text{not Id}"] \\
x \arrow[r, mapsto, "\theta(g)"] & gx
\end{tikzcd}$$

🟥 $x \mapsto rx$

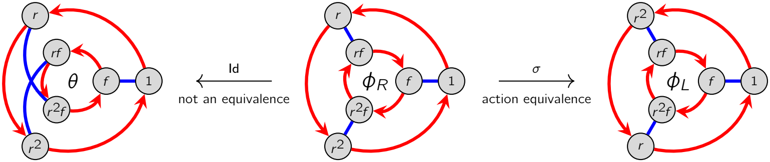
🟦 $x \mapsto fx$

🟥 $x \mapsto xr$

🟦 $x \mapsto xf$

🟥 $x \mapsto r^{-1}x = r^2x$

🟦 $x \mapsto f^{-1}x = fx$



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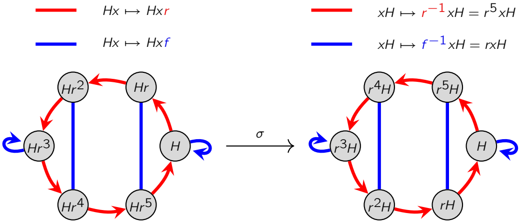
cosets by multiplication	$Hx \mapsto Hxg$	$xH \mapsto g^{-1}xH$

Recall that $aH = bH$ implies $Ha^{-1} = Hb^{-1}$.

$$\begin{tikzcd}
Hx \arrow[r, mapsto, "\phi_R(g)"] \arrow[d, "\sigma"'] & Hxg \arrow[d, "\sigma"] \\
x^{-1}H \arrow[r, mapsto, dashed, "\phi_L(g)"] & g^{-1}x^{-1}H
\end{tikzcd}$$

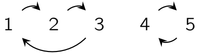


Since $aH = bH \not\Rightarrow Ha = Hb$, the the map $xH \mapsto Hx$ is not even well-defined.

Actions by permutations matrices

Consider the following permutation $\pi \in S_5$:

i	1	2	3	4	5
$\pi(i)$	2	3	1	5	4



$$\pi = (1\,2\,3)\,(4\,5)$$

The permutation matrix P_π permutes the entries of a column vector as

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_1 \\ x_2 \\ x_5 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_{\pi^{-1}(1)} \\ x_{\pi^{-1}(2)} \\ x_{\pi^{-1}(3)} \\ x_{\pi^{-1}(4)} \\ x_{\pi^{-1}(5)} \end{bmatrix},$$

and the entries of a row vector as

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} x_2 & x_3 & x_1 & x_5 & x_4 \end{bmatrix}$$

$$= \begin{bmatrix} x_{\pi(1)} & x_{\pi(2)} & x_{\pi(3)} & x_{\pi(4)} & x_{\pi(5)} \end{bmatrix}.$$

Actions by permutations matrices

In general, a left action of S_n on a set of vectors X

$$\phi_L \colon S_n \longrightarrow \mathrm{Perm}(X), \qquad \phi_L(\pi) \colon x \longmapsto P_\pi x$$

is equivalent to the right action

$$\phi_R \colon S_n \longrightarrow \mathrm{Perm}(X), \qquad \phi_R(\pi) \colon x \longmapsto x^T P_\pi^T = x^T P_{\pi^{-1}}$$

via the transpose map.

$$\begin{tikzcd}
x \arrow[r, mapsto, "\phi_L(\pi)"] \arrow[d, mapsto, "T"'] & P_\pi x \arrow[d, mapsto, "T"] \\
x^T \arrow[r, mapsto, dashed, "\phi_R(\pi)"] & x^T P_{\pi^{-1}}
\end{tikzcd}$$

$$\begin{tikzcd}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \arrow[r, mapsto, "P_\pi = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}"] \arrow[d, mapsto, "T"'] & \begin{bmatrix} x_3 \\ x_1 \\ x_2 \end{bmatrix} \arrow[d, mapsto, "T"] \\
\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \arrow[r, mapsto, dashed, "P_{\pi^{-1}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = P^T"] & \begin{bmatrix} x_3 & x_1 & x_2 \end{bmatrix}
\end{tikzcd}$$

Another equilalence between left and right actions of permutations

Recall the two "canonical" ways label a Cayley graph for $S_3 = \langle (12), (23) \rangle$ with the set

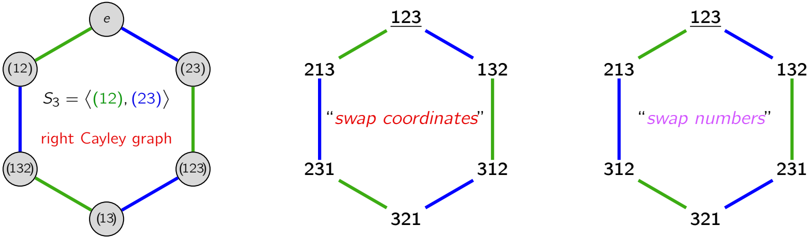
$$X = \{\mathbf{123}, \mathbf{132}, \mathbf{213}, \mathbf{231}, \mathbf{312}, \mathbf{321}\}.$$

In one, (ij) can be interpreted to mean

"*swap the numbers in the* i^{th} *and* j^{th} *coordinates*."

Alternatively, (ij) could mean

"*swap the numbers* i *and* j*, regardless of where they are*."



One of these is a right group action, and the other a left group action

Another equilalence between left and right actions of permutations

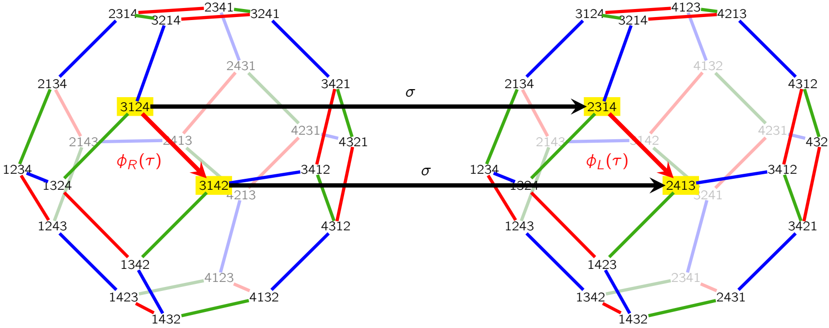
i	1	2	3	4
π	**3**	**1**	**2**	**4**

i	1	2	3	4
$\pi\tau$	**3**	**1**	**4**	**2**

$$\begin{tikzcd}
\pi(1)\pi(2)\pi(3)\pi(4) \arrow[r, mapsto, "\phi_R(\tau)", "\textit{swap coordinates 3 and 4}"'] \arrow[d, mapsto, "\sigma"'] & \pi(1)\pi(2)\pi(4)\pi(3) \arrow[d, mapsto, "\sigma"] \\
\pi^{-1}(1)\pi^{-1}(2)\pi^{-1}(3)\pi^{-1}(4) \arrow[r, mapsto, "\phi_L(\tau)", "\textit{swap digits 3 and 4}"'] & \pi^{-1}(1)\pi^{-1}(2)\pi^{-1}(4)\pi^{-1}(3)
\end{tikzcd}$$

i	1	2	3	4
π^{-1}	**2**	**3**	**1**	**4**

i	1	2	3	4
$\tau^{-1}\pi^{-1}$	**2**	**4**	**1**	**3**



"*swap coordinates*"   "*swap numbers*"