Visual Algebra

Lecture 5.7: Transitive actions

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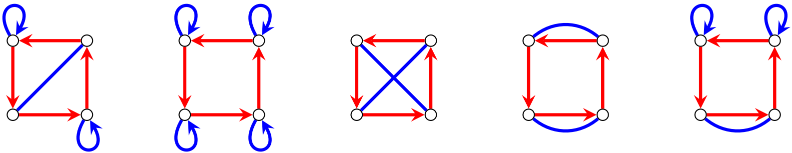
Classification of G-sets

Natural question

Given a group G, what are its possible (connected) G-sets?

For example, which of the following can arise as an orbit of an action by $G = D_4$?



Definition

An action $\phi \colon G \to \mathsf{Perm}(S)$, and the G-set S, is

- transitive if it has only one orbit: ("*graph is connected*")
- free if $\mathsf{stab}(s) = \langle e \rangle$ for all $s \in S$. ("*uncollapsed – no nontrivial loops*")
- faithful if $\mathsf{Ker}(\phi) = \langle e \rangle$. ("*no broken buttons, except* $1 \in G$")

In this language our question becomes: "*classify all transitive G-actions*" (or G-sets).

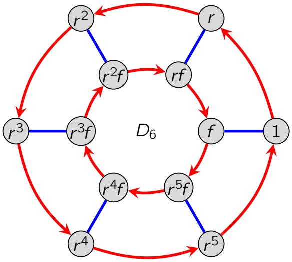
Transitive actions

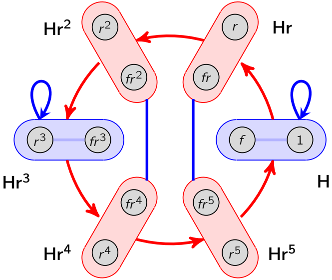
Let's say that two G-actions are isomorphic if the corresponding G-sets are isomorphic.

Proposition

Every transitive G-action is isomorphic to G acting on a set of cosets by multiplication.

A connected action graph is a Cayley graph collapsed by right cosets of some subgroup.

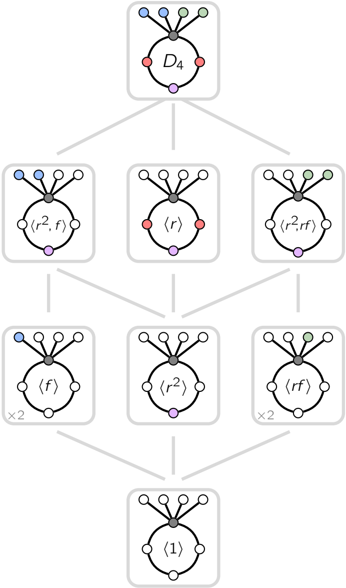


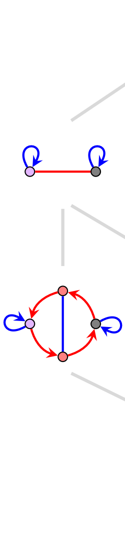


collapse right cosets of H (an action)

We can *always* collapse by right cosets. We can collapse by left cosets iff H is normal.

The transitive D_4-sets: collapsing by right cosets





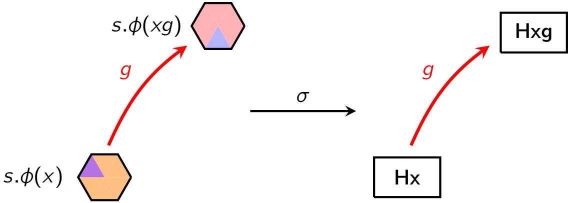
Transitive actions

Proposition

Every transitive G-action is isomorphic to G acting on a set of cosets by multiplication.

Proof sketch. Let $\iota\colon G \to G$ be the identity, fix $s \in S$, let $H = \mathbf{stab}(s)$, and define

$$\sigma\colon S \longrightarrow H\backslash G, \qquad \sigma\colon s.\phi(x) \longmapsto Hx$$



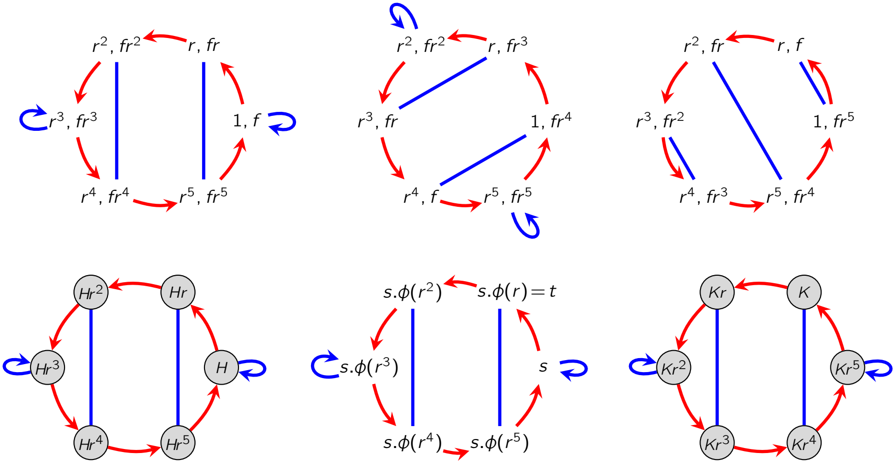
Show that σ is a well-defined bijection, and then the proof follows because:

$$\begin{array}{ccc} S & \xrightarrow{\ \phi(g)\ } & S \\ \sigma\downarrow & & \downarrow\sigma \\ H\backslash G & \xrightarrow[\ \psi(g)\ ]{} & H\backslash G \end{array} \qquad\qquad \begin{array}{ccc} s.\phi(x) & \xrightarrow{\ \phi(g)\ } & s.\phi(xg) \\ \sigma\downarrow & & \downarrow\sigma \\ Hx & \xmapsto[\ \psi(g)\ ]{} & Hxg \end{array}$$

Conjugates of stab(s) give the same G-set

Proposition

If $K = a^{-1}Ha$, then $H\backslash G$ and $K\backslash G$ are isomorphic G-sets.

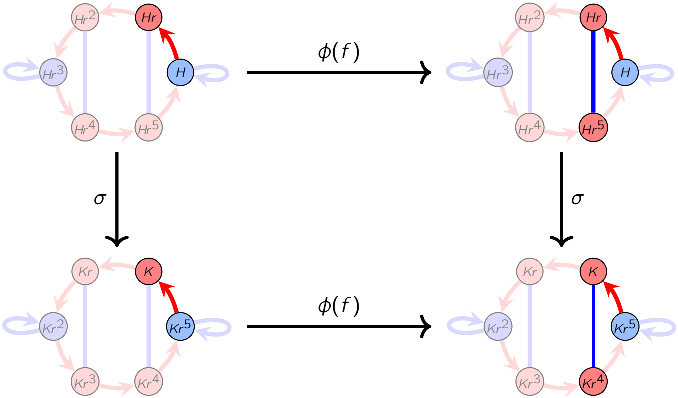


Conjugates of stab(s) give the same G-set

Proposition

If $K = a^{-1}Ha$, then $H\backslash G$ and $K\backslash G$ are isomorphic G-sets.

Consider $H = \langle f \rangle$ and $K = r^{-1}Hr = \langle r^4 f \rangle$. Define $\sigma \colon Hx \mapsto Kr^{-1}x$.



Conjugates of stab(s) give the same G-set

Proposition

If $K = a^{-1}Ha$, then $H\backslash G$ and $K\backslash G$ are isomorphic G-sets.

Proof

Define the map

$$\sigma\colon H\backslash G \longrightarrow K\backslash G, \qquad \sigma\colon Hx \longmapsto Ka^{-1}x.$$

We claim that this is a well-defined bijection, and commutes with $\phi(g)$:

$$\begin{array}{ccc} H\backslash G & \xrightarrow{\phi(g)} & H\backslash G \\ \big\downarrow{\scriptstyle\sigma} & & \big\downarrow{\scriptstyle\sigma} \\ K\backslash G & \xrightarrow[\phi(g)]{} & K\backslash G \end{array} \qquad\qquad \begin{array}{ccc} Hx & \xrightarrow{\phi(g)} & Hxg \\ \big\downarrow{\scriptstyle\sigma} & & \big\downarrow{\scriptstyle\sigma} \\ Ka^{-1}x & \xmapsto[\phi(g)]{} & Ka^{-1}xg \end{array}$$

<u>Well-defined</u>: Suppose $Hx = Hy$. Then $Hyx^{-1} = H$, so $yx^{-1} \in H$.

$$\sigma(Hx) = Ka^{-1}x = \underbrace{a^{-1}H}_{=Ka^{-1}}x = a^{-1}\underbrace{(Hyx^{-1})}_{=H}x = \underbrace{a^{-1}H}_{=Ka^{-1}}y = Ka^{-1}y = \sigma(Hy).$$

Conjugates of stab(s) give the same G-set

Proposition

If $K = a^{-1}Ha$, then the G-sets $H\backslash G$ and $K\backslash G$ are isomorphic.

Proof

Define the map

$$\sigma\colon H\backslash G \longrightarrow K\backslash G, \qquad \sigma\colon Hx \longmapsto Ka^{-1}x.$$

We claim that this is a well-defined bijection, and commutes with $\phi(g)$:

$$\begin{array}{ccc} H\backslash G & \xrightarrow{\phi(g)} & H\backslash G \\ \sigma\downarrow & & \downarrow\sigma \\ K\backslash G & \xrightarrow[\phi(g)]{} & K\backslash G \end{array} \qquad\qquad \begin{array}{ccc} Hx & \xrightarrow{\phi(g)} & Hxg \\ \sigma\downarrow & & \downarrow\sigma \\ Ka^{-1}x & \xmapsto[\phi(g)]{} & Ka^{-1}xg \end{array}$$

Injectivity: Suppose $\sigma(Hx) = \sigma(Hy)$. Then

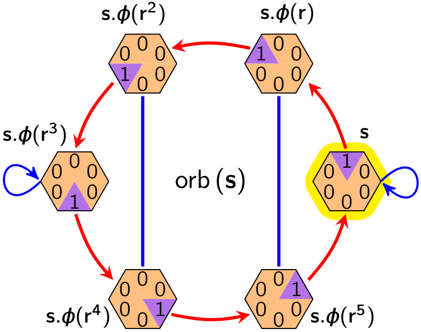
$$\sigma(Hx) = \underbrace{Ka^{-1}}_{=a^{-1}H}x = a^{-1}Hx, \qquad \text{and} \qquad \sigma(Hy) = \underbrace{Ka^{-1}}_{=a^{-1}H}y = a^{-1}Hy,$$

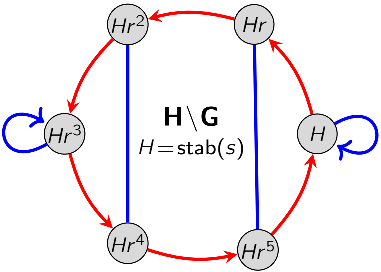
and thus $Hx = Hy$. Surjectivity is straightforward. $\square$

Transitive actions

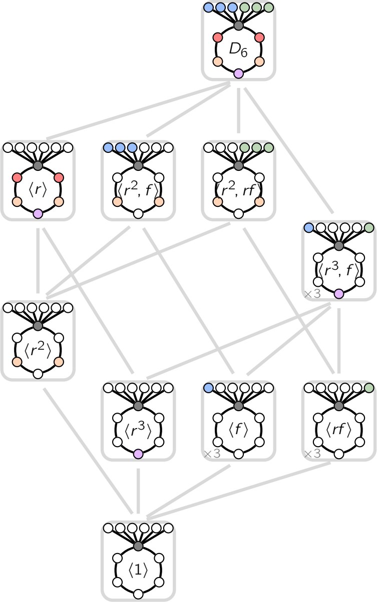
Big ideas

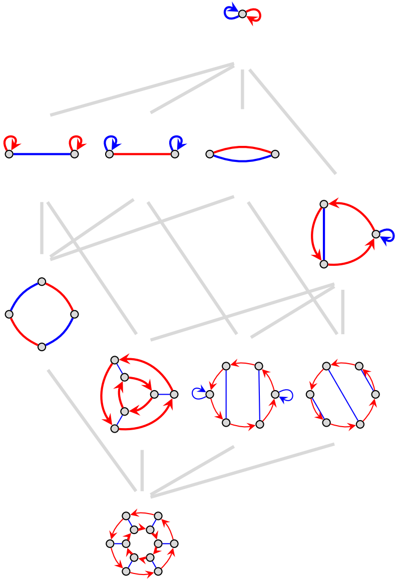
- Every transitive G-action is isomorphic to G acting on the cosets of $\mathbf{stab}(s)$.
- The action graph is constructed by collapsing by right cosets of $\mathbf{stab}(s)$.
- conjugates of $\mathbf{stab}(s)$ give the same G-set.





The transitive D_6-sets: collapsing by right cosets





Subgroups of small index

Groups acting on cosets is a useful technique for establishing seemingly unrelated results.

Several of these involve showing that subgroups of "small index" are normal.

We've already seen that subgroups of index 2 are normal.

Of course, there are non-normal index-3 subgroups, like $\langle f \rangle \leq D_3$.

The following gives a sufficient condition for when index-3 subgroups are normal.

Proposition

If G has no subgroup of index 2, then any subgroup of index 3 is normal.

Proof

Let $H \leq G$ with $[G : H] = 3$.

Let G act on the cosets of H by multiplication, to get a nontrivial homomorphism

$$\phi\colon G \longrightarrow S_3.$$

$K := \mathrm{Ker}(\phi) \leq H$ is the largest normal subgroup of G contained in H. By the FHT,

$$G/K \cong \mathrm{Im}(\phi) \leq S_3.$$

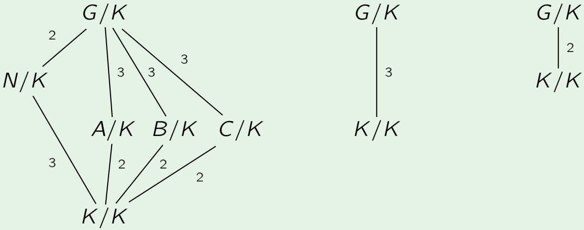
Subgroups of small index

Proof (contin.)

Thus, there are three cases for this quotient:

$$G/K \cong S_3, \qquad G/K \cong C_3, \qquad G/K \cong C_2.$$

Visually, this means that we have one of the following:



By the corrdespondence theorem, $K \leq H \lneq G$ implies $K/K \leq H/K \lneq G/K$.

Since G has no index-2 subgroup, only the middle case is possible (*Why?*).

This forces $K/K = H/K$, and so $K = H$, which is normal for multiple reasons. □

Subgroups of small index

Proposition

Suppose $H \leq G$ and $[G : H] = p$, the smallest prime dividing $|G|$. Then $H \trianglelefteq G$.

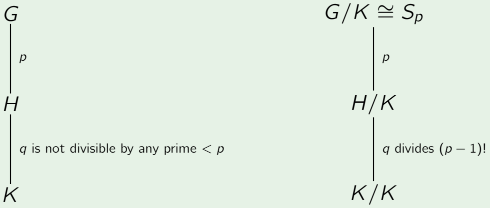
Proof

Let G act on the cosets of H by multiplication, to get a non-trivial homomorphism

$$\phi \colon G \longrightarrow S_p.$$

The kernel $K = \mathrm{Ker}(\phi)$, is the largest normal subgroup of G such that $K \leq H \lneq G$.

We'll show that $H = K$, or equivalently, that $[H : K] = 1$. By the correspondence theorem:



Do you see why $q = 1$? $\square$