Visual Algebra

Lecture 5.10: Normalizers of *p*-subgroups

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A creative application of a group action

Cauchy's theorem

If p is a prime dividing $|G|$, then G has an element (and hence a subgroup) of order p.

Proof

Let P be the set of ordered p-tuples of elements from G whose product is e:

$$(x_1, x_2, \dots, x_p) \in P \quad \text{iff} \quad x_1 x_2 \cdots x_p = e\,.$$

Observe that $|P| = |G|^{p-1}$. (We can choose $x_1, \dots, x_{p-1}$ freely; then x_p is forced.)

The group $\mathbb{Z}_p$ acts on P by cyclic shift:

$$\phi\colon \mathbb{Z}_p \longrightarrow \mathrm{Perm}(P), \qquad (x_1, x_2, \dots, x_p) \stackrel{\phi(1)}{\longmapsto} (x_2, x_3 \dots, x_p, x_1)\,.$$

The set P is partitioned into orbits, each of size $|\,\mathrm{orb}(s)| = [\mathbb{Z}_p : \mathrm{stab}(s)] = 1$ or p.

The only way that the orbit of $(x_1, x_2, \dots, x_p)$ can have size 1 is if $x_1 = \cdots = x_p$.

Clearly, $(e, \dots, e) \in P$ is a fixed point.

The $|G|^{p-1} - 1$ other elements in P sit in orbits of size 1 or p.

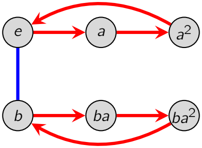
Since $p \nmid |G|^{p-1} - 1$, there must be other orbits of size 1. Thus, some $(x, \dots, x) \in P$, with $x \neq e$ satisfies $x^p = e$. □

Classification of groups of order 6

By Cauchy's theorem, every group of order 6 must have:

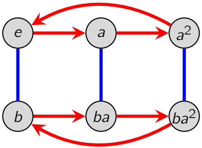
- an element a of order 3
- an element b of order 2.

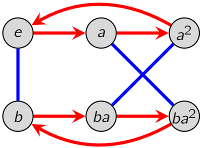
Clearly, $G = \langle a, b \rangle$, and so G must have the following "partial Cayley graph":



It is now easy to see that up to isomorphism, there are only 2 groups of order 6:

$C_6 \cong C_2 \times C_3$





D_3

Exercise. Classify groups of order 8 with a similar argument.

p-groups and the Sylow theorems

Definition

A p-group is a group whose order is a power of a prime p. A p-group that is a subgroup of a group G is a p-subgroup of G.

Notational convention

Throughout, G will be a group of order $|G| = p^n \cdot m$, with $p \nmid m$. That is, p^n is the *highest power* of p dividing $|G|$.

There are three Sylow theorems, and loosely speaking, they describe the following about a group's p-subgroups:

1. **Existence**: In every group, p-subgroups of all possible sizes exist.
2. **Relationship**: All maximal p-subgroups are conjugate.
3. **Number**: Strong restrictions on the number of p-subgroups a group can have.

Together, these place strong restrictions on the structure of a group G with a fixed order.

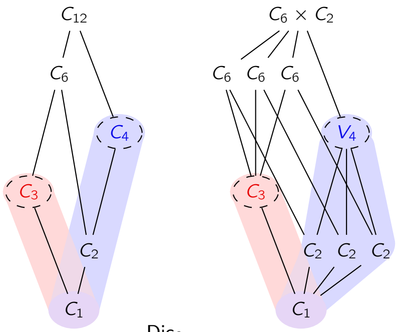
The five groups of order 12

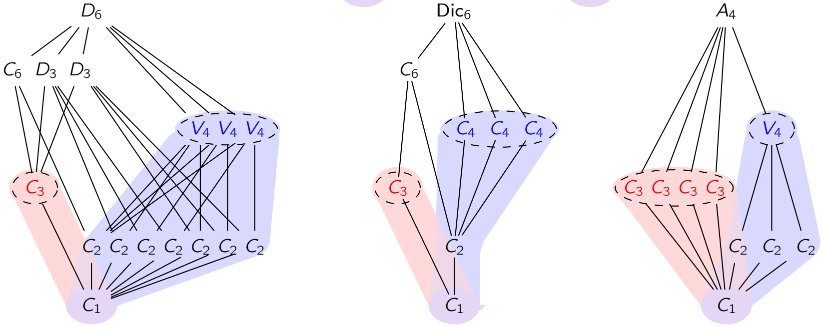
Sylow theorems:

p-subgroups come in "towers."

2-subgroups are blue

3-subgroups are red.





Normalizers of p-subgroups

Before we introduce the Sylow theorems, we need to better understand p-groups.

Recall that a p-group is any group of order p^n. Examples, of 2-groups that we've seen include C_1, C_4, V_4, D_4 and Q_8, C_8, $C_4 \times C_2$, D_8, SD_8, Q_{16}, SA_8, DQ_8,...

p-group Lemma

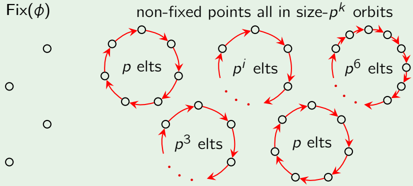
If a p-group G acts on a set S via $\phi\colon G \to \text{Perm}(S)$, then

$$|\operatorname{Fix}(\phi)| \equiv_p |S|.$$

Proof (sketch)

Suppose $|G| = p^n$.

By the orbit-stabilizer theorem, the only possible orbit sizes are $1, p, p^2, \dots, p^n$.



Normalizers of p-subgroups

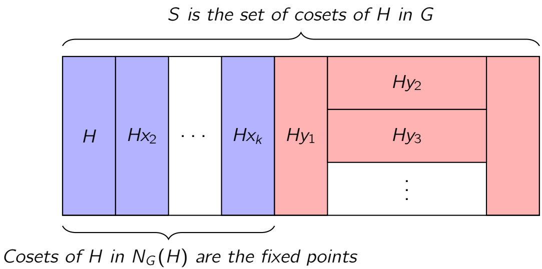
Normalizer lemma, Part 1

If H is a p-subgroup of G, then

$$[N_G(H) \colon H] \equiv_p [G \colon H] .$$

Approach:

- Let H (not G!) act on the (right) cosets of H by (right) multiplication.



- Apply our lemma: $|\operatorname{Fix}(\phi)| \equiv_p |S|$.

Proof of the Normalizer lemma

Normalizer lemma, Part 1

If H is a p-subgroup of G, then

$$[N_G(H) \colon H] \equiv_p [G \colon H] \,.$$

Proof

Let $S = H\backslash G = \{Hx \mid x \in G\}$. The group H acts on S by **right-multiplication**, via $\phi \colon H \to \mathsf{Perm}(S)$, where

$$\phi(h) = \text{the permutation sending each } Hx \text{ to } Hxh.$$

The fixed points of ϕ are the cosets Hx in the normalizer $N_G(H)$:

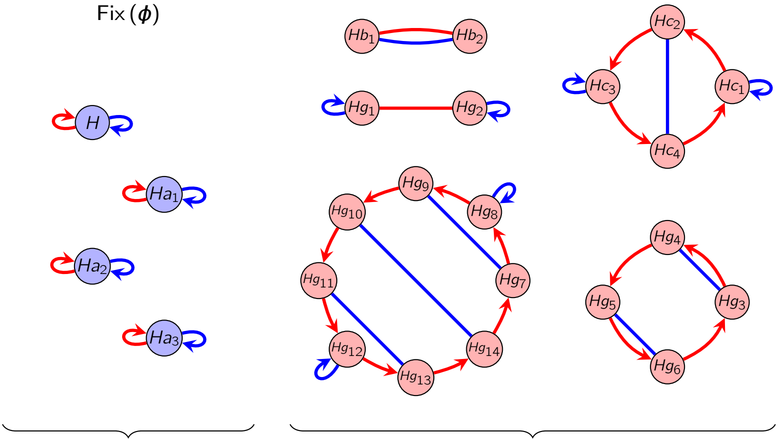
$$\begin{aligned} Hxh = Hx, \quad \forall h \in H \qquad &\Longleftrightarrow \qquad Hxhx^{-1} = H, \quad \forall h \in H \\ &\Longleftrightarrow \qquad xhx^{-1} \in H, \quad \forall h \in H \\ &\Longleftrightarrow \qquad x \in N_G(H)\,. \end{aligned}$$

Therefore, $|\operatorname{Fix}(\phi)| = [N_G(H) \colon H]$, and $|S| = [G : H]$. By our p-group Lemma,

$$|\operatorname{Fix}(\phi)| \equiv_p |S| \quad \Longrightarrow \quad [N_G(H) \colon H] \equiv_p [G \colon H]. \qquad \square$$

Normalizers of p-subgroups

Here is a picture of the action of the p-subgroup H (for $p = 2$) on the set $S = H\backslash G$, from the proof of the normalizer lemma.



The fixed points are the cosets in $N_G(H)$

Cosets not in $N_G(H)$ are in orbits of order p^i, for various $i \geq 1$

Normalizers of p-subgroups

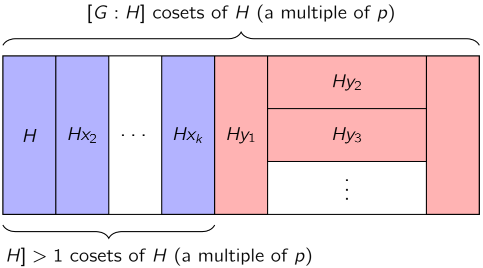
Recall that $H \leq N_G(H)$ (always), and H is fully unnormal if $H = N_G(H)$.

Normalizer lemma, Part 2

Suppose $|G| = p^n m$, and $H \leq G$ with $|H| = p^i < p^n$. Then $H \lneq N_G(H)$, and the index $[N_G(H) : H]$ is a multiple of p.

H is not "*fully unnormal*":

$$H \lneq N_G(H) \leq G$$



Important corollaries

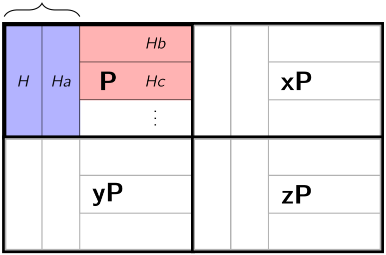
- p-groups cannot have any fully unnormal subgroups (i.e., $H \lneq N_G(H)$).
- In *any* finite group, the only fully unnormal p-subgroups are maximal.

Normalizers of p-subgroups

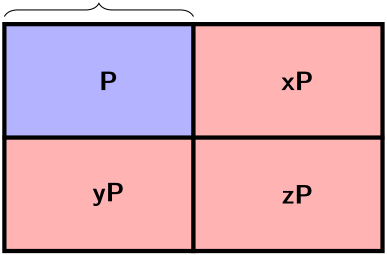
Let H be properly contained in a maximal p-subgroup $P \lneq G$.

- The normalizer of H *must* grow in P (and hence in G)
- The normalizer of P *need not* grow in G.

$H \lneq N_P(H) \leq N_G(H)$



it may happen that $P = N_G(P)$



Proof of the normalizer lemma

Normalizer lemma, Part 2

Suppose $|G| = p^n m$, and $H \leq G$ with $|H| = p^i < p^n$. Then $H \lneq N_G(H)$, and the index $[N_G(H) : H]$ is a multiple of p.

Proof

Since $H \trianglelefteq N_G(H)$, we can create the quotient map

$$\pi \colon N_G(H) \longrightarrow N_G(H)/H\,, \qquad\qquad \pi \colon g \longmapsto gH\,.$$

The size of the quotient group is $[N_G(H) \colon H]$, the number of cosets of H in $N_G(H)$.

By the normalizer lemma Part 1, $[N_G(H) \colon H] \equiv_p [G \colon H]$. By Lagrange's theorem,

$$[N_G(H) \colon H] \equiv_p [G \colon H] = \frac{|G|}{|H|} = \frac{p^n m}{p^i} = p^{n-i} m \equiv_p 0.$$

Therefore, $[N_G(H) \colon H]$ is a multiple of p, so $N_G(H)$ must be strictly larger than H. □