Visual Algebra

Lecture 5.11: The first two Sylow theorems

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The Sylow theorems

Recall the following question that we asked earlier in this course.

Open-ended question

What group structural properties are possible, what are impossible, and how does this depend on $|G|$?

One approach is to decompose large groups into "building block subgroups." For example:

> *given a group of order* $72 = 2^3 \cdot 3^2$*, what can we say about its* 2*-subgroups and* 3*-subgroups?.*

This is the idea behind the Sylow theorems, developed by Norwegian mathematician Peter Sylow (1832–1918).

The Sylow theorems address the following questions of a finite group G:

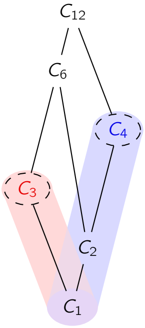
1. How big are its p-subgroups?
2. How are the p-subgroups related?
3. How many p-subgroups are there?
4. What can we say about their conjugacy classes?

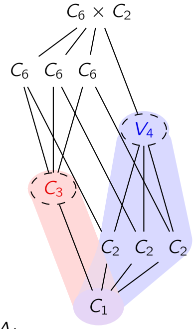
An example: groups of order 12

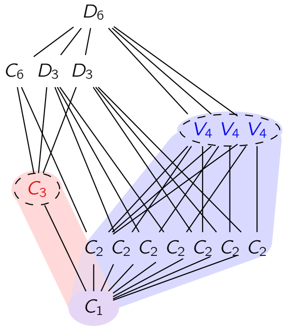
The Sylow theorems can be used to classify all groups of order 12.

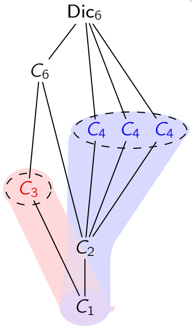
We've already seen them all.

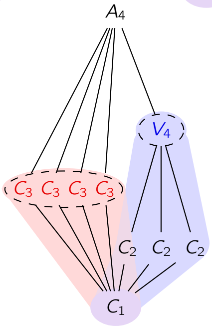
What patterns do you notice about the 2-groups and 3-groups, that might generalize to all p-subgroups?











The Sylow theorems

Notational convention

Througout, G will be a group of order $|G| = p^n \cdot m$, with $p \nmid m$.

That is, p^n is the *highest power* of p dividing $|G|$.

A subgroup of order p^n is called a Sylow p-subgroup.

Let $\text{Syl}_p(G)$ denote the set of Sylow p-subgroups, and $n_p := \big|\text{Syl}_p(G)\big|$.

There are three Sylow theorems, and loosely speaking, they describe the following about a group's p-subgroups:

1. **Existence**: In every group, p-subgroups of all possible sizes exist, and they're "*nested*".
2. **Relationship**: All maximal ("Sylow") p-subgroups are conjugate.
3. **Number**: There are strong restrictions on n_p, the number of Sylow p-subgroups.

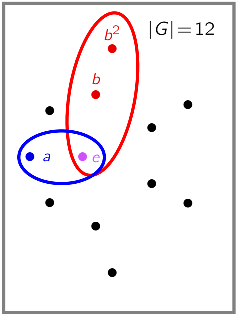
Together, these place strong restrictions on the structure of a group G with a fixed order.

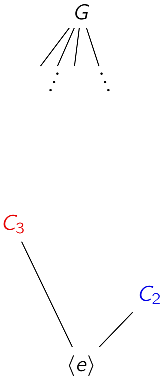
Our unknown group of order 12

Throughout, we will have a running example, a "mystery group" G of order $12 = 2^2 \cdot 3$.

We already know a little bit about G. By Cauchy's theorem, it must have:

- an element a of order 2, and
- an element b of order 3.





Using *only* the fact that $|G| = 12$, we will unconver as much about its structure as we can.

The 1st Sylow theorem: existence of p-subgroups

First Sylow theorem

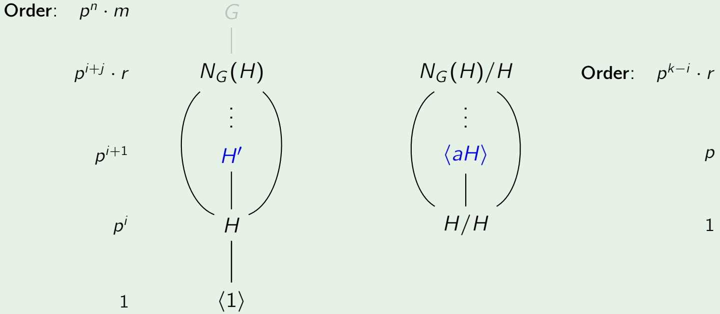
G has a subgroup of order p^k, for each p^k dividing $|G|$.

Also, every non-Sylow p-subgroup sits inside a larger p-subgroup.

Proof

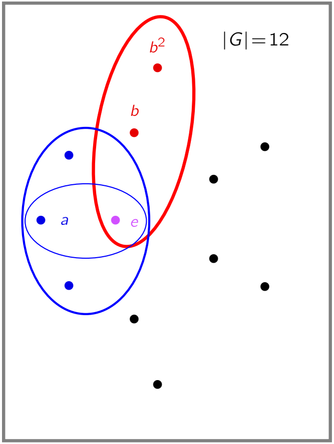
Take any $H \leq G$ with $|H| = p^i < p^n$. We know $H \trianglelefteq N_G(H)$ and p divides $|N_G(H)/H|$.

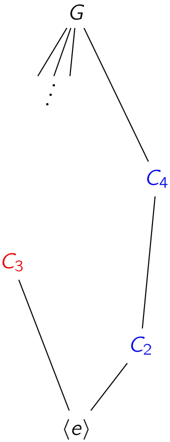
Find an element aH of order p. The union of cosets in $\langle aH \rangle$ is a subgroup of order p^{i+1}.

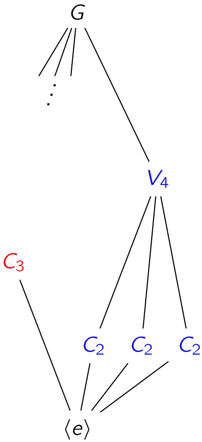


Our unknown group of order 12

By the first Sylow theorem, $\langle a \rangle$ is contained in a subgroup of order 4, which could be V_4 or C_4, or possibly both.







The 2^{nd} Sylow theorem: relationship among p-subgroups

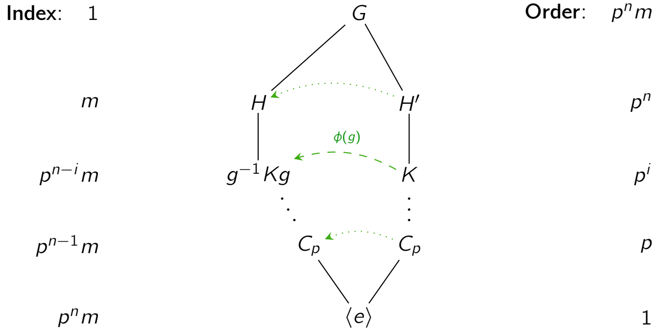
Second Sylow theorem

Any two Sylow p-subgroups are conjugate (and hence isomorphic).

We'll actually prove a stronger version, which easily implies the 2nd Sylow theorem.

Strong second Sylow theorem

Let $H \in \text{Syl}(G)$, and $K \leq G$ any p-subgroup. Then K is conjugate to a subgroup of H.



The 2^{nd} Sylow theorem: All Sylow p-subgroups are conjugate

Strong second Sylow theorem

Let H be a Sylow p-subgroup, and $K \leq G$ any p-subgroup. Then K is conjugate to some subgroup of H.

Proof

Let $S = H\backslash G = \{Hg \mid g \in G\}$, the set of right cosets of H.

The group K acts on S by **right-multiplication**, via $\phi\colon K \to \mathsf{Perm}(S)$, where

$$\phi(k) = \text{the permutation sending each } Hg \text{ to } Hgk.$$

A fixed point of ϕ is a coset $Hg \in S$ such that

$$\begin{aligned} Hgk = Hg\,, \quad \forall k \in K \qquad &\Longleftrightarrow \qquad Hgkg^{-1} = H\,, \quad \forall k \in K \\ &\Longleftrightarrow \qquad gkg^{-1} \in H\,, \quad \forall k \in K \\ &\Longleftrightarrow \qquad gKg^{-1} \subseteq H. \end{aligned}$$

Thus, *if we can show that ϕ has a fixed point Hg, we're done!*

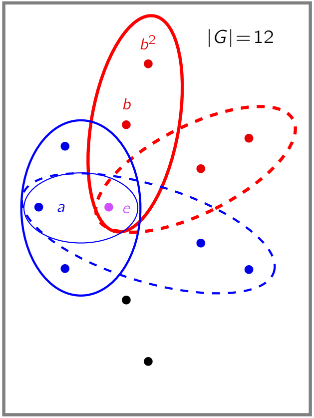
All we need to do is show that $|\operatorname{Fix}(\phi)| \not\equiv_p 0$. By the p-group Lemma,

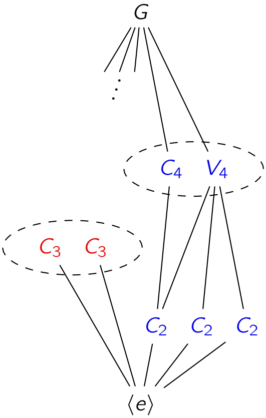
$$|\operatorname{Fix}(\phi)| \equiv_p |S| = [G : H] = m \not\equiv_p 0. \qquad \square$$

Our unknown group of order 12

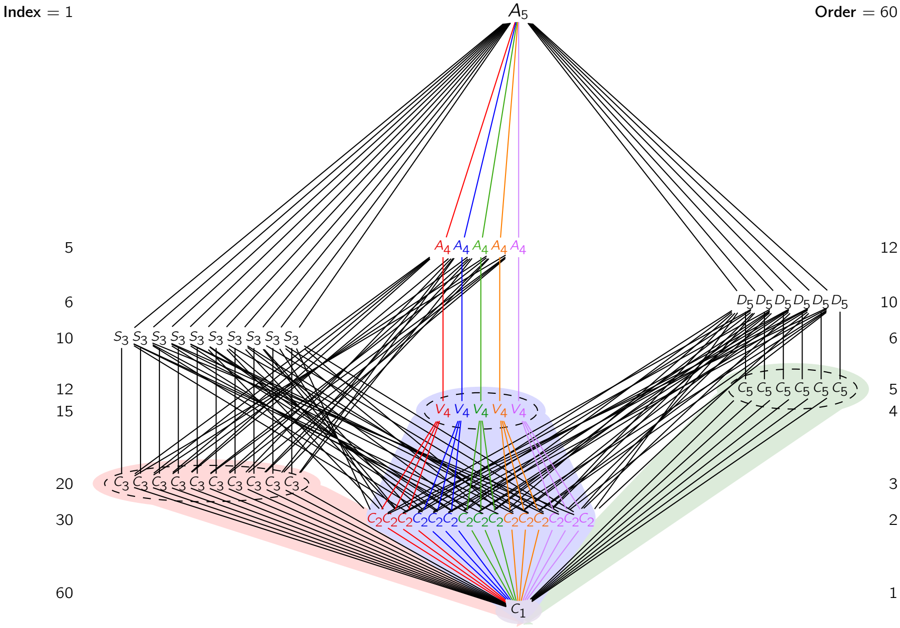
By the second Sylow theorem, all Sylow p-subgroups are conjugate, and hence isomorphic.

This eliminates the following subgroup lattice of a group of order 12.





Example: A_5 has no nontrival proper normal subgroups



The normalizer of the normalizer

Notice how in A_5:

- all Sylow p-subgroups are moderately unnormal
- the normalizer of each Sylow p-subgroup is fully unnormal. That is:

$$N_G(N_G(P)) = N_G(P)$$

Proposition

Let P be a non-normal Sylow p-subgroup of G. Then its normalizer is fully unnormal.

Proof

We'll verify the equivalent statement of $N_G(N_G(P)) = N_G(P)$.

Note that P is a normal Sylow p-subgroup of $N_G(P)$.

By the 2nd Sylow theorem, P is the unique Sylow p-subgroup of $N_G(P)$.

Take an element x that normalizes $N_G(P)$ (i.e., $x \in N_G(N_G(P))$. We'll show that it also normalizes P. By definition, $xN_G(P)x^{-1} = N_G(P)$, and so

$$P \leq N_G(P) \qquad \Longrightarrow \qquad xPx^{-1} \leq xN_G(P)x^{-1} = N_G(P).$$

But xPx^{-1} is also a Sylow p-subgroup of $N_G(P)$, and by uniqueness, $xPx^{-1} = P$. □