Visual Algebra

Lecture 5.12: The third Sylow theorem and simple groups

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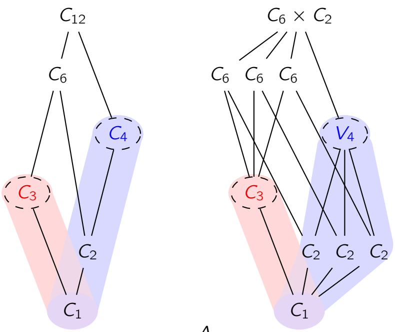
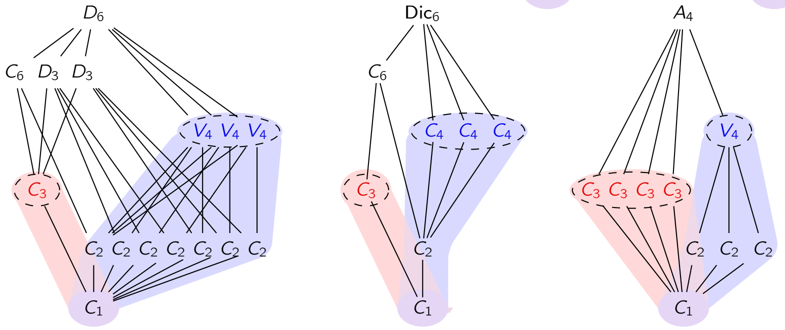
Summary of the Sylow theorems

1st **Sylow theorem**: p-subgroups come in "towers."

2nd **Sylow theorem**: the tops of the p-group towers are conjugate.

3rd **Sylow theorem**: characterizes the possible sizes of these conjugacy classes.





The 3^{rd} Sylow theorem: number of Sylow p-subgroups

Third Sylow theorem

Let n_p be the number of Sylow p-subgroups of G. Then

$$n_p \text{ divides } |G| \qquad \text{and} \qquad n_p \equiv_p 1\,.$$

(Note that together, these imply that $n_p \mid m$, where $|G| = p^n \cdot m$.)

Proof

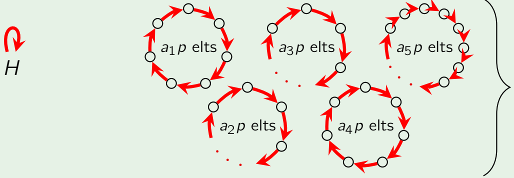
Take $H \in \mathrm{Syl}_p(G)$. By the 2nd Sylow theorem, $n_p = |\,\mathrm{cl}_G(H)| = [G : N_G(H)] \,\big|\, |G|$. ✓

The subgroup H acts on $S = \mathrm{Syl}_p(G)$ by conjugation, via $\phi\colon G \to \mathrm{Perm}(S)$, where

$$\phi(h) = \text{the permutation sending each } K \text{ to } h^{-1}Kh.$$

Goal: *show that H is the unique fixed point*.

$|\,\mathrm{Fix}(\phi)| = 1$ *other Sylow p-subgroups are in larger orbits*



total # Sylow p-subgroups $= n_p = |S| \equiv_p |\,\mathrm{Fix}(\phi)|$

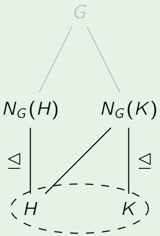
Proof (cont.)

Goal: *show that H is the unique fixed point*.

Let $K \in \mathrm{Fix}(\phi)$. Then $K \leq G$ is a Sylow p-subgroup satisfying

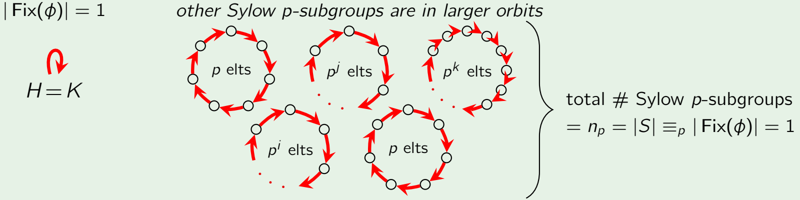
$$h^{-1}Kh = K\,, \quad \forall h \in H \quad \Longleftrightarrow \quad H \leq N_G(K) \leq G\,.$$

- H and K are p-Sylow in G, and in $N_G(K)$.
- H and K are conjugate in $N_G(K)$. (2nd Sylow thm.)
- $K \trianglelefteq N_G(K)$, thus is only conjugate to itself in $N_G(K)$.



Thus, $K = H$. That is, $\mathrm{Fix}(\phi) = \{H\}$.

By the p-group Lemma, $n_p := |S| \equiv_p |\mathrm{Fix}(\phi)| = 1$. □



Summary of the proofs of the Sylow theorems

For the 1st Sylow theorem, we started with $H = \{e\}$, and inductively created larger subgroups of size $p, p^2, \dots, p^n$.

For the 2^{nd} and 3^{rd} Sylow theorems, we used a clever group action and then applied one or both of the following:

(i) *orbit-stabilizer theorem*. If G acts on S, then $|\operatorname{orb}(s)| \cdot |\operatorname{stab}(s)| = |G|$.

(ii) *p-group lemma*. If a p-group acts on S, then $|S| \equiv_p |\operatorname{Fix}(\phi)|$.

To summarize, we used:

S2 The action of $K \in \operatorname{Syl}_p(G)$ on $S = H \backslash G$ by right multiplication for some other $H \in \operatorname{Syl}_p(G)$.

S3a The action of G on $S = \operatorname{Syl}_p(G)$, by conjugation.

S3b The action of $H \in \operatorname{Syl}_p(G)$ on $S = \operatorname{Syl}_p(G)$, by conjugation.

Our mystery group order 12

By the 3rd Sylow theorem, every group G of order $12 = 2^2 \cdot 3$ must have:

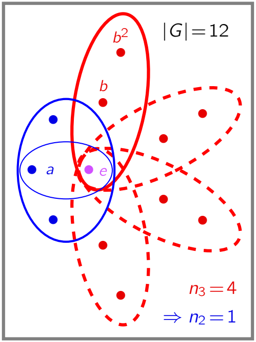
- n_3 Sylow 3-subgroups, each of order 3.

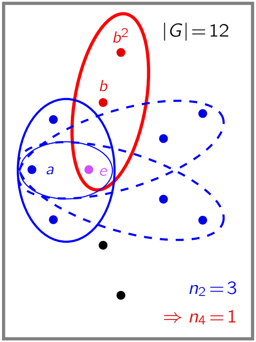
$$n_3 \mid 4, \qquad n_3 \equiv 1 \pmod 3 \qquad \Longrightarrow \qquad n_3 = 1 \text{ or } 4.$$

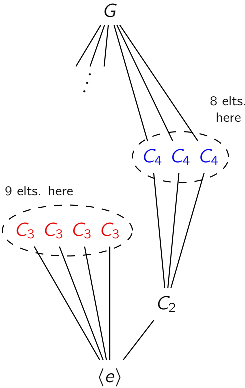
- n_2 Sylow 2-subgroups of order $2^2 = 4$.

$$n_2 \mid 3, \qquad n_2 \equiv 1 \pmod 2 \qquad \Longrightarrow \qquad n_2 = 1 \text{ or } 3.$$

But both are not possible! (There aren't enough elements.)



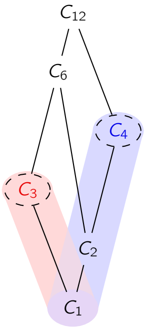


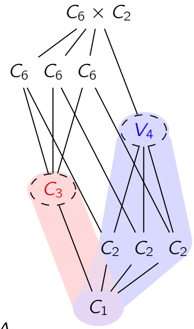


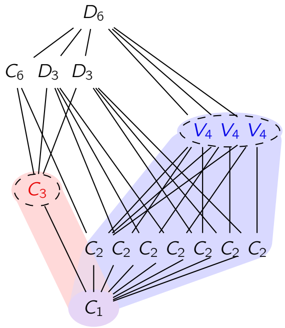
Classification of groups of order 12

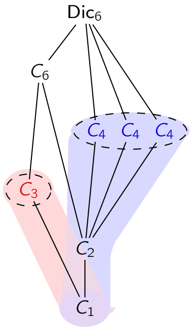
$P = C_4$ or V_4, $Q = C_3$

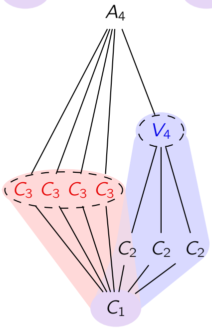
- **Case 1**: $n_2 = 1$, $n_3 = 1$.
- **Case 2**: $n_2 = 1$, $n_3 = 4$.
- **Case 3**: $n_2 = 3$, $n_3 = 1$.











Simple groups and the Sylow theorems

Definition

A group G is simple if its only normal subgroups are G and $\langle e \rangle$.

Simple groups are to groups what primes are to integers, and are essential to understand.

The Sylow theorems are very useful for establishing statements like:

"There are no simple groups of order k (for some k)."

Since all Sylow p-subgroups are conjugate, the following result is immediate.

Remark

A Sylow p-subgroup is normal in G iff it's the unique Sylow p-subgroup (that is, if $n_p = 1$).

Thus, if we can show that $n_p = 1$ for some p dividing $|G|$, then G cannot be simple.

For some $|G|$, this is harder than for others, and sometimes it's not possible.

Tip

When trying to show that $n_p = 1$, it's usually helpful to analyze the largest primes first.

An easy example

We'll see three examples of showing that groups of a certain size cannot be simple, in successive order of difficulty.

Then we'll see several that I will leave as exercises.

Proposition

There are no simple groups of order 84.

Proof

Since $|G| = 84 = 2^2 \cdot 3 \cdot 7$, the third Sylow theorem tells us:

- n_7 divides $2^2 \cdot 3 = 12$ (so $n_7 \in \{1, 2, 3, 4, 6, 12\}$)
- $n_7 \equiv_7 1$.

The only possibility is that $n_7 = 1$, so the Sylow 7-subgroup must be normal. □

Observe why it is beneficial to use the largest prime first:

- n_3 divides $2^2 \cdot 7 = 28$ and $n_3 \equiv_3 1$. Thus $n_3 \in \{1, \not{2}, 4, 7, \not{14}, 28\}$.
- n_2 divides $3 \cdot 7 = 21$ and $n_2 \equiv_2 1$. Thus $n_2 \in \{1, 3, 7, 21\}$.

A harder example

Proposition

There are no simple groups of order 351.

Proof

Since $|G| = 351 = 3^3 \cdot 13$, the third Sylow theorem tells us:

- n_{13} divides $3^3 = 27$ (so $n_{13} \in \{1, 3, 9, 27\}$)
- $n_{13} \equiv_{13} 1$.

The only possibilies are $n_{13} = 1$ or 27.

A Sylow 13-subgroup P has order 13, and a Sylow 3-subgroup Q has order $3^3 = 27$. Therefore, $P \cap Q = \{e\}$.

Suppose $n_{13} = 27$. Every Sylow 13-subgroup contains 12 non-identity elements, and so G must contain $27 \cdot 12 = 324$ elements of order 13.

This leaves $351 - 324 = 27$ elements in G not of order 13. Thus, G contains only one Sylow 3-subgroup (i.e., $n_3 = 1$) and so G cannot be simple. □

Another example

Proposition

There are no simple groups of order $24 = 2^3 \cdot 3$.

From the 3rd Sylow theorem, we can only conclude that $n_2 \in \{1, 3\}$ and $n_3 = \{1, 4\}$.

Let H be a Sylow 2-subgroup, which has relatively "small" index: $[G : H] = 3$.

Lemma

If G has a subgroup of index $[G : H] = n$, and $|G|$ does not divide $n!$, then G is not simple.

Proof

Let G act on the **right cosets** of H (i.e., $S = H\backslash G$) by **right-multiplication**:

$$\phi\colon G \longrightarrow \text{Perm}(S) \cong S_n\,, \qquad \phi(g) = \text{the permutation that sends each } Hx \text{ to } Hxg.$$

Recall that $\text{Ker}(\phi) \trianglelefteq G$, and is the intersection of all conjugate subgroups of H:

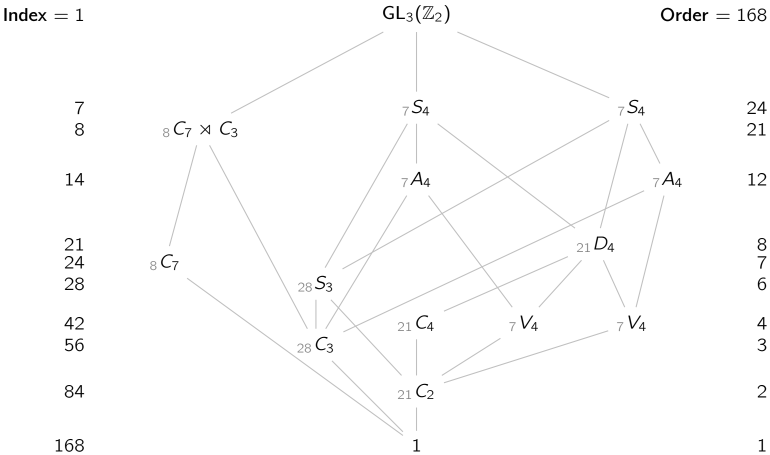
$$\langle e \rangle \leq \text{Ker}(\phi) = \bigcap_{x \in G} x^{-1}Hx \lneq G$$

If $\text{Ker}(\phi) = \langle e \rangle$ then $\phi\colon G \hookrightarrow S_n$ is an embedding, which is impossible because $|G| \nmid n!$. $\square$

The second smallest non-abelian simple group

Exercise

Show that the simple group $G = \mathsf{GL}_3(\mathbb{Z}_2)$ of order 168 is a subgroup of A_8.



A_6: the third smallest non-abelian simple group

Exercise

Prove that there are no simple groups of order $90 = 2 \cdot 3^2 \cdot 5$.