Visual Algebra

Lecture 6.9: Central series and extensions

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Chapter overview

Chemistry investigates how matter is assembled from basic "*building blocks*" (atoms).

Main goal

Understand how groups are assembled from basic "building blocks" (simple groups).

This chapter is broken into three parts; this lecture is on Part 3(e):

1. Finite abelian groups are products of cyclic groups.
2. The classification of finite simple groups: the "*periodic table of groups*."
3. Extensions of groups: like doing "*all of chemistry* for groups."
   - (a) Groups built from a (right) split extension (semidirect products)
   - (b) Groups built from a left split extension (direct products)
   - (c) Groups built from simple extensions (all groups)
   - (d) Groups built from abelian extensions (solvable groups)
   - (e) **Groups built from central extensions (nilpotent groups)**

Central series

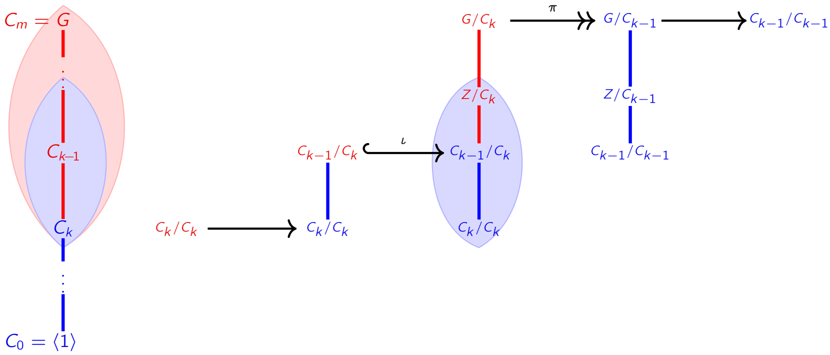
Definition

A **central series** of a group G is a normal series

$$G = C_0 \trianglerighteq C_1 \trianglerighteq \cdots \trianglerighteq C_m = \langle 1 \rangle, \qquad \text{such that } C_{k-1}/C_k \leq Z(G/C_k).$$

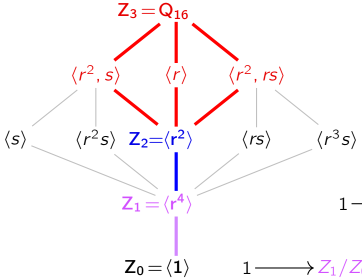
Equivalently, G/C_k **is a central extension of** G/C_{k-1} **by** C_{k-1}/C_k.

$$1 \longrightarrow C_{k-1}/C_k \xrightarrow{\iota_k} G/C_k \xrightarrow{\pi_k} G/C_{k-1} \longrightarrow 1$$



Key idea

Climb up a central series to construct a nilpotent group with central extensions.



$$
\begin{tikzcd}
 & & & 1 \arrow[r] & G/G \arrow[r, hook] & G/G \arrow[r, two heads] & C_1 \arrow[r] & 1 \\
 & & 1 \arrow[r] & G/Z_2 \arrow[r, hook] & G/Z_2 \arrow[r, two heads] & G/G \arrow[r] \arrow[u, dashed] & 1 & \\
 & 1 \arrow[r] & Z_2/Z_1 \arrow[r, hook] & G/Z_1 \arrow[r, two heads] & G/Z_2 \arrow[r] \arrow[u, dashed] & 1 & & \\
1 \arrow[r] & Z_1/Z_0 \arrow[r, hook] & G/Z_0 \arrow[r, two heads] & G/Z_1 \arrow[r] \arrow[u, dashed] & 1 & & &
\end{tikzcd}
$$

We can say:

- Q_{16} is a central extension of $D_4 \cong Q_{16}/Z_1$ by $Z(Q_{16}) \cong C_2$,...
- ...and D_4 is a central extension of $V_4 \cong Q_{16}/Z_2$ by $Z(D_4) \cong C_2$,...
- ...and V_4 is a central extension of $C_1 \cong Q_{16}/Z_3$ by $Z(V_4) \cong V_4$.

Nilpotency and central extensions

$$\begin{tikzcd} & & 1 \arrow[r] & C_2 \arrow[r, hook] & \mathbf{D_4} \arrow[r, two heads] & V_4 \arrow[r] & 1 \\ 1 \arrow[r] & C_2 \arrow[r, hook] & \mathbf{Q_{16}} \arrow[r, two heads] & D_4 \arrow[r] \arrow[ur, dashed] & 1 \end{tikzcd}$$

"*D_4 is constructible with 1 central extension*"

"*Q_{16} is constructible with 2 central extensions*"

Definition

Say that G is constructible with ...

- ... 1 central extension if for some abelian A_1 and Q_0,

$$1 \longrightarrow A_1 \hookrightarrow G \twoheadrightarrow Q_0 \longrightarrow 1$$

- ... k central extensions if

$$1 \longrightarrow A_k \hookrightarrow G \twoheadrightarrow Q_{k-1} \longrightarrow 1$$

where $\iota(A_k)$ is central, and Q_{k-1} is constructible with $k-1$ central extensions.

Nilpotency and central extensions

If G is constructible with k central extensions, then we have the following:

$$
\begin{tikzcd}
 & & & & 1 \arrow[r] & G/G \arrow[r, hook] & G/C_0 \arrow[r, two heads] & 1 \arrow[r] & 1 \\
 & & & 1 \arrow[r] & C_0/C_1 \arrow[r, hook] & G/C_1 \arrow[r, two heads] & G/C_0 \arrow[r] \arrow[u, dashed] & 1 & \\
 & & 1 \arrow[r] & C_1/C_2 \arrow[r, hook] & G/C_2 \arrow[r, two heads] & G/C_1 \arrow[r] \arrow[u, dashed] & 1 & & \\
 & 1 \arrow[r] & C_{k-2}/C_{k-1} \arrow[r, hook] & G/C_{k-1} \arrow[r, two heads] & G/C_{k-2} \arrow[r] \arrow[u, dashed] & 1 & & & \\
1 \arrow[r] & C_{k-1}/C_k \arrow[r, hook] & G/C_k \arrow[r, two heads] & G/C_{k-1} \arrow[r] \arrow[u, dashed] & 1 & & & &
\end{tikzcd}
$$

Note that G has a normal central series

$$G = C_0 \trianglerighteq \cdots \trianglerighteq C_k = \langle 1 \rangle, \qquad \text{each } C_{i-1}/C_i \text{ is central in } G/C_i, \text{ and } C_i \trianglelefteq G.$$

Conversely, given a normal central series, we can reconstruct the exact sequences.

Central series

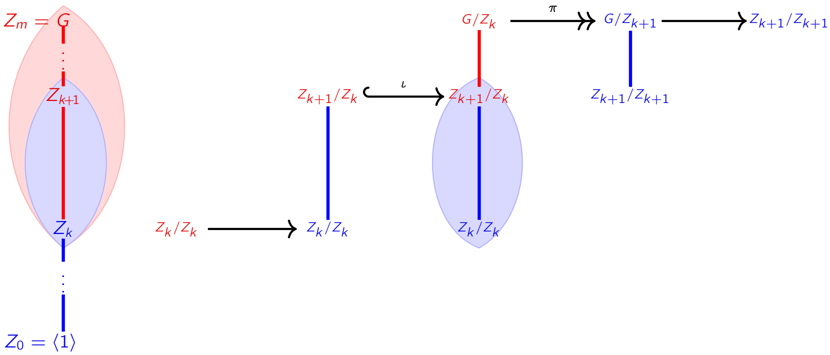
Remark

The **ascending central series** of a nilpotent group G is a normal series

$$\langle 1\rangle = Z_0 \trianglelefteq Z_1 \trianglelefteq \cdots \trianglelefteq Z_m = G, \quad \text{such that} \quad Z_{k+1}/Z_k = Z(G/Z_k).$$

Equivalently, G/Z_k **is the maximal central extension of** G/Z_{k+1} (by Z_{k+1}/Z_k).

$$1 \longrightarrow Z_{k+1}/Z_k \xrightarrow{\iota_k} G/Z_k \xrightarrow{\pi_k} G/Z_{k+1} \longrightarrow 1$$



Central series

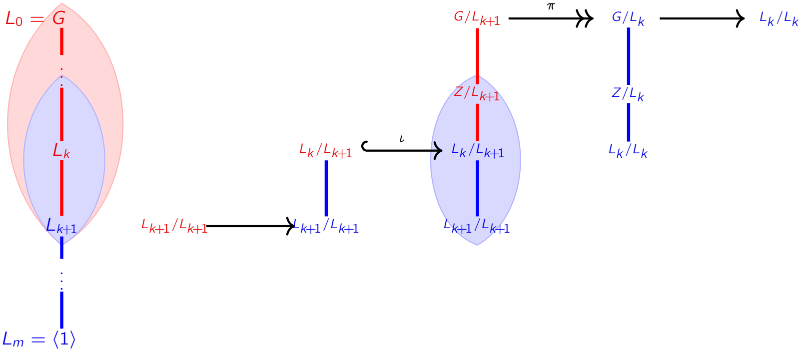
Remark

The **descending central series** of a group G is a normal series

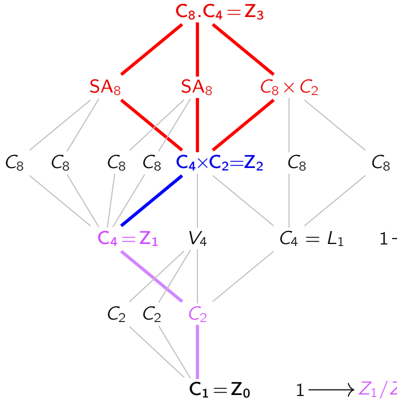
$$G = L_0 \trianglerighteq L_1 \trianglerighteq \cdots \trianglerighteq L_m = G, \quad \text{such that} \quad L_k/L_{k+1} \leq Z(G/L_{k+1}).$$

Equivalently, G/L_{k+1} **is a central extension of** G/C_k **by** L_k/L_{k+1}.

$$1 \longrightarrow L_k/L_{k+1} \xrightarrow{\iota_k} G/L_{k+1} \xrightarrow{\pi_k} G/L_k \longrightarrow 1$$



The ascending central series of $G = C_8.C_4$ (GAP ID 32.15)



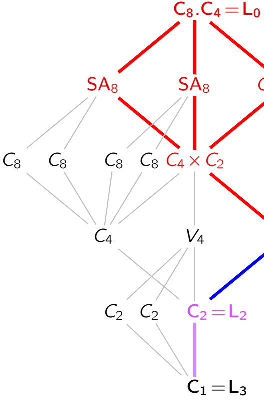
$$\begin{tikzcd}
 & & & 1 \arrow[r] & G/G \arrow[r, hook] & G/Z_3 \arrow[r, two heads] & C_1 \arrow[r] & 1 \\
 & & 1 \arrow[r] & Z_3/Z_2 \arrow[r, hook] & G/Z_2 \arrow[r, two heads] & G/Z_3 \arrow[r] \arrow[u, dashed] & 1 & \\
 & 1 \arrow[r] & Z_2/Z_1 \arrow[r, hook] & G/Z_1 \arrow[r, two heads] & G/Z_2 \arrow[r] \arrow[u, dashed] & 1 & & \\
1 \arrow[r] & Z_1/Z_0 \arrow[r, hook] & G/Z_0 \arrow[r, two heads] & G/Z_1 \arrow[r] \arrow[u, dashed] & 1 & & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & 1 \arrow[r] & C_2 \arrow[r, hook] & \mathbf{D_4} \arrow[r, two heads] & V_4 \arrow[r] & 1 \\
1 \arrow[r] & C_4 \arrow[r, hook] & \mathbf{C_8.C_4} \arrow[r, two heads] & D_4 \arrow[r] \arrow[u, dashed] & 1 &
\end{tikzcd}$$

"*D_4 is constructible with 1 central extension*"

"*$C_8.C_4$ is constructible with 2 central extensions*"

The descending central series of $G = C_8.C_4$ (GAP ID 32.15)



$$
\begin{tikzcd}[column sep=small]
 & & & 1 \arrow[r] & G/G \arrow[r, hook] & G/L_0 \arrow[r, two heads] & C_1 \arrow[r] & 1 \\
 & & 1 \arrow[r] & L_0/L_1 \arrow[r, hook] & G/L_1 \arrow[r, two heads] & G/L_0 \arrow[r] \arrow[u, dashed] & 1 & \\
 & 1 \arrow[r] & L_1/L_2 \arrow[r, hook] & G/L_2 \arrow[r, two heads] & G/L_1 \arrow[r] \arrow[u, dashed] & 1 & & \\
1 \arrow[r] & L_2/L_3 \arrow[r, hook] & G/L_3 \arrow[r, two heads] & G/L_2 \arrow[r] \arrow[u, dashed] & 1 & & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & 1 \arrow[r] & C_2 \arrow[r, hook] & \mathbf{C_4 \rtimes C_4} \arrow[r, two heads] & C_4 \rtimes C_4 \arrow[r] & 1 \\
1 \arrow[r] & C_2 \arrow[r, hook] & \mathbf{C_8.C_4} \arrow[r, two heads] & C_4 \times C_2 \arrow[r] \arrow[u, dashed] & 1 &
\end{tikzcd}
$$

"$C_4 \rtimes C_4$ *is constructible with 1 central extension*"

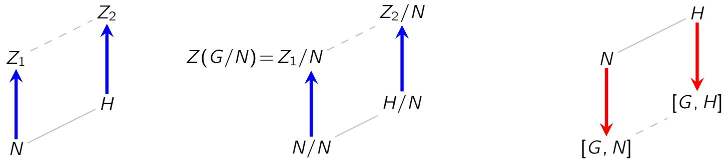
"$C_8.C_4$ *is constructible with 2 central extensions*"

Monotonicity of central ascents and descents

Proposition

Let $N \leq H \leq G$ be a chain of normal subgroups. Then

1. If $Z(G/N) = Z_1/N$ and $Z(G/H) = Z_2/H$, then $Z_1 \leq Z_2$.
2. $[G, N] \leq [G, H]$.



Proof of (i)

For any $z \in Z_1$, the coset zN is central in G/N, which means that, for all $g \in G$,

$$\begin{aligned}
zNgN = gNzN \quad &\Longleftrightarrow \quad [z, g] \leq N && \textit{by the central ascent lemma} \\
&\Longrightarrow \quad [z, g] \leq H && \textit{by assumption, } N \leq H \\
&\Longleftrightarrow \quad zHgH = gHzH && \textit{by the central ascent lemma} \\
&\Longleftrightarrow \quad zH \in Z(G/H) && \textit{by definition of } Z(G/H) \\
&\Longleftrightarrow \quad z \in Z_2 && \textit{by definition; } Z(G/H) = Z_2/H.
\end{aligned}$$

The crooked ladder theorem

Let G be a finite group, and suppose that either of the following hold:

1. The descending central series reaches the bottom: $L_{n-1} \gneq L_n = \langle 1 \rangle$.
2. The ascending central series reaches the top: $Z_{n-1} \lneq Z_n = G$.

Then for all $k = 0, \dots, n$,

$$L_{n-k} \leq Z_k.$$

maximal central descents

$$
\begin{tikzcd}
L_0 \arrow[d] & = G = & \mathbf{Z_n} \\
L_1 \arrow[d] \arrow[rr, dash] & & \mathbf{Z_{n-1}} \arrow[u] \\
L_2 \arrow[d] \arrow[rr, dash] & & \mathbf{Z_{n-2}} \arrow[u] \\
\vdots \arrow[d] & & \vdots \arrow[u] \\
L_{n-2} \arrow[d] \arrow[rr, dash] & & \mathbf{Z_2} \arrow[u] \\
L_{n-1} \arrow[d] \arrow[rr, dash] & & \mathbf{Z_1} \arrow[u] \\
L_n & \leq & \mathbf{Z_0} \arrow[u] & = \langle 1 \rangle
\end{tikzcd}
$$

maximal central ascents

The crooked ladder theorem

Let G be a finite group, and suppose that either of the following hold:

(i) The descending central series reaches the bottom: $L_{n-1} \gneq L_n = \langle 1 \rangle$.

(ii) The ascending central series reaches the top: $Z_{n-1} \lneq Z_n = G$.

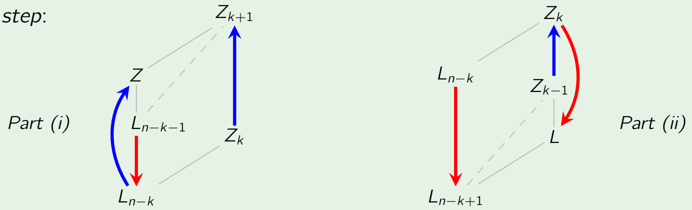
Then for all $k = 0, \dots, n$,

$$L_{n-k} \leq Z_k.$$

Proof of (i); Part (ii) is analogous (HW)

Induct on k. The base case is trivial: $L_n = Z_0 = \langle 1 \rangle$.

Inductive step:



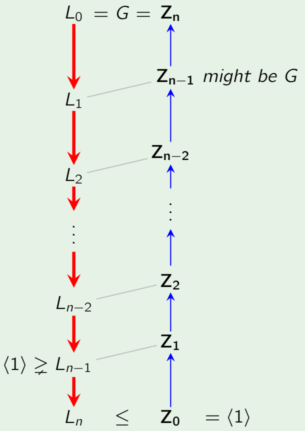
Note that L_{n-k-1} is a central ascent from L_{n-k}: $\overbrace{L_{n-k} \leq L_{n-k-1} \leq Z}^{L_{n+k-1}/L_{n-k} \in Z(G/L_{n-k})} \leq Z_{k+1}$, where $\underbrace{Z \leq Z_{k+1}}_{\text{monotonicity}}$. $\square$

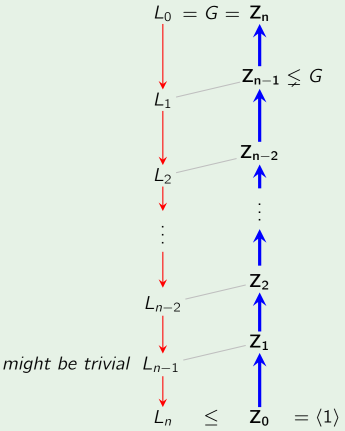
The ascending and descending central series have the same length

Corollary

The ascending central series reaches $Z_n = G$ iff the descending central series reaches $L_m = \langle 1 \rangle$. If this happens, their lengths are the same.

Proof



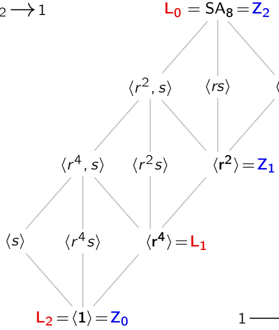


Ascending vs. descending central series

Here's a familiar example, higlighting the "crooked ladder property,"

$$L_{n-k} \leq Z_k, \qquad \textit{or equivalently,} \quad L_k \leq Z_{n-k}.$$

$$\begin{tikzcd}
 & & 1 \arrow[r] & L_1 \arrow[r, hook] \arrow[d, dashed] & \mathbf{L_0} \arrow[r, two heads] & C_4 \times C_2 \arrow[r] & 1 \\
 & 1 \arrow[r] & L_2 \arrow[r, hook] \arrow[d, dashed] & \mathbf{L_1} \arrow[r, two heads] & C_2 \arrow[r] & 1 & \\
1 \arrow[r] & 1 \arrow[r, hook] & \mathbf{L_2} \arrow[r, two heads] & 1 \arrow[r] & 1 & &
\end{tikzcd}$$

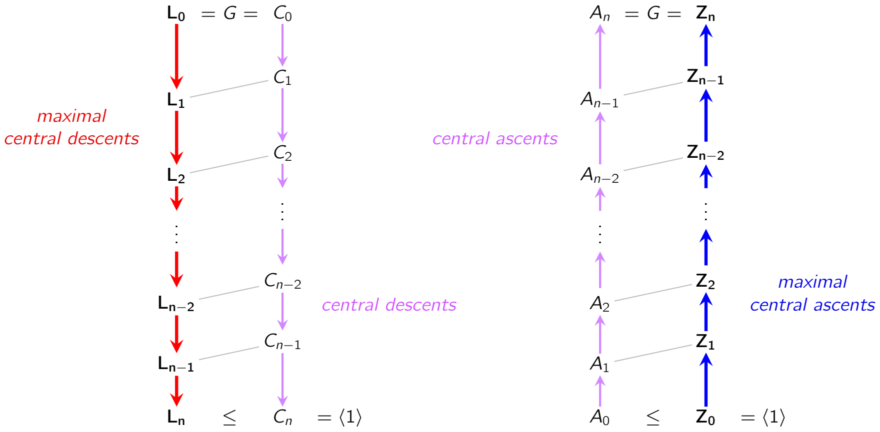


$$\begin{tikzcd}
 & & 1 \arrow[r] & C_4 \arrow[r, hook] & \mathbf{Z_2} \arrow[r, two heads] & V_4 \arrow[r] & 1 \\
 & 1 \arrow[r] & 1 \arrow[r, hook] & \mathbf{Z_1} \arrow[r, two heads] \arrow[u, dashed] & C_4 \arrow[r] & 1 & \\
1 \arrow[r] & 1 \arrow[r, hook] & \mathbf{Z_0} \arrow[r, two heads] \arrow[u, dashed] & 1 \arrow[r] & 1 & &
\end{tikzcd}$$

Also known as the "upper" and "lower" central series

Aside (exercise)

- The L_k's fall faster than every other central series, and thus are term-by-term lower.
- The Z_k's rise faster than every other central series, and thus are term-by-term higher.



Solvability and nilpotency in terms of extensions

Summary

- **Every finite group** can be constructed from **extensions of simple groups**.
- Solvable groups can be constructed from abelian extensions.
- Nilpotent groups can be constructed from central extensions.