Visual Algebra

Lecture 6.10: Characterizations of nilpotent groups

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Chapter overview

Chemistry investigates how matter is assembled from basic "*building blocks*" (atoms).

Main goal

Understand how groups are assembled from basic "building blocks" (simple groups).

This chapter is broken into three parts; this lecture is on Part 3(e):

1. Finite abelian groups are products of cyclic groups.
2. The classification of finite simple groups: the "*periodic table of groups*."
3. Extensions of groups: like doing "*all of chemistry* for groups."
   - (a) Groups built from a (right) split extension (semidirect products)
   - (b) Groups built from a left split extension (direct products)
   - (c) Groups built from simple extensions (all groups)
   - (d) Groups built from abelian extensions (solvable groups)
   - (e) **Groups built from central extensions (nilpotent groups)**

The first two characterizations of nilpotency

Original definition

A finite group G is **nilpotent** if the ascending central series reaches the top of the lattice:

$$\langle 1\rangle = Z_0 \trianglelefteq Z_1 \trianglelefteq \cdots \trianglelefteq Z_m = G, \qquad \text{where} \qquad Z_{k+1}/Z_k = Z(G/Z_k).$$

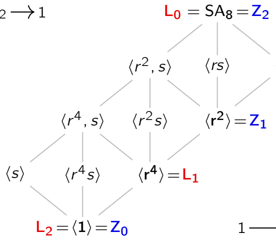
Theorem

The ascending central series reachers the top of the lattice iff the descending central series

$$G = L_0 \trianglerighteq L_1 \trianglerighteq \cdots \trianglerighteq L_m = \langle 1\rangle, \qquad L_{k+1} = [G, L_k]$$

reaches the bottom of the lattice. If this happens, it takes the same number of steps.

$$\begin{tikzcd}
 & & 1 \arrow[r] & L_1 \arrow[r, hook] \arrow[d, dashed] & \mathbf{L_0} \arrow[r, two heads] & C_4\times C_2 \arrow[r] & 1 \\
 & 1 \arrow[r] & L_2 \arrow[r, hook] \arrow[d, dashed] & \mathbf{L_1} \arrow[r, two heads] & C_2 \arrow[r] & 1 & \\
1 \arrow[r] & 1 \arrow[r, hook] & \mathbf{L_2} \arrow[r, two heads] & 1 \arrow[r] & 1 & &
\end{tikzcd}$$



$$\begin{tikzcd}
 & & 1 \arrow[r] & C_4 \arrow[r, hook] & \mathbf{Z_2} \arrow[r, two heads] & V_4 \arrow[r] & 1 \\
 & 1 \arrow[r] & 1 \arrow[r, hook] & \mathbf{Z_1} \arrow[r, two heads] \arrow[u, dashed] & C_4 \arrow[r] & 1 & \\
1 \arrow[r] & 1 \arrow[r, hook] & \mathbf{Z_0} \arrow[r, two heads] \arrow[u, dashed] & 1 \arrow[r] & 1 & &
\end{tikzcd}$$

The 3$^{\text{rd}}$ characterization of nilpotency

Lemma

A finite groups is nilpotent if and only if it is constructible with central extensions.

This is equivalent to G having a central series:

$$G = C_0 \trianglerighteq C_1 \trianglerighteq \cdots \trianglerighteq C_m = \langle 1 \rangle, \qquad \text{such that } C_{k-1}/C_k \leq Z(G/C_k).$$

$$\begin{tikzcd}
 & & & & 1 \arrow[r] & G/G \arrow[r, hook] & G/C_0 \arrow[r, two heads] & 1 \arrow[r] & 1 \\
 & & & 1 \arrow[r] & C_0/C_1 \arrow[r, hook] & G/C_1 \arrow[r, two heads] & G/C_0 \arrow[r] \arrow[u, dashed] & 1 \\
 & & 1 \arrow[r] & C_1/C_2 \arrow[r, hook] & G/C_2 \arrow[r, two heads] & G/C_1 \arrow[r] \arrow[u, dashed] & 1 \\
 & 1 \arrow[r] & C_{k-2}/C_{k-1} \arrow[r, hook] & G/C_{k-1} \arrow[r, two heads] & G/C_{k-2} \arrow[r] \arrow[u, dashed] & 1 \\
1 \arrow[r] & C_{k-1}/C_k \arrow[r, hook] & G/C_k \arrow[r, two heads] & G/C_{k-1} \arrow[r] \arrow[u, dashed] & 1
\end{tikzcd}$$

Products of nilpotent groups are nilpotent

Lemma

If $G = H \times K$, then $L_n(G) = L_n(H) \times L_n(K)$ for all n.

Proof

The proof is by induction. The base case is easy:

$$G = L_0(G) = L_0(H) \times L_0(K) = H \times K.$$

Next, suppose that $L_k(G) = L_k(H) \times L_k(K)$. Then

$$\begin{aligned} L_{k+1}(G) = [H \times K, L_k(H \times K)] &= [H \times K, L_k(H) \times L_k(K)] \\ &= [H, L_k(H)] \times [K, L_k(K)] \\ &= L_{k+1}(H) \times L_{k+1}(K), \end{aligned}$$

and the result follows inductively.

Corollary

If H and K are nilpotent, then so is $G = H \times K$.

The 4^{th} characterization of nilpotency: normalizers grow

Proposition

A finite group G is nilpotent iff it has no proper fully unnormal subgroups ($H \lneq N_G(H)$).

Proof

"$\Rightarrow$" Take the maximal Z_k containing H. We'll show $N_G(H)$ contains Z_{k+1}.

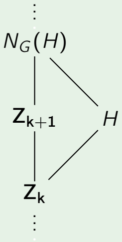
Pick some $x \in Z_{k+1}$. (Need to show it normalizes H.)

For all $g \in G$, we have $[x, g] \in Z_k$.

Thus, $[x, h] = xhx^{-1}h^{-1} \in Z_k \leq H, \quad$ for all $h \in H$.

Since $xhx^{-1}h^{-1} \in H$, then $xhx^{-1} \in H$.

Thus, $x \in N_G(H)$.



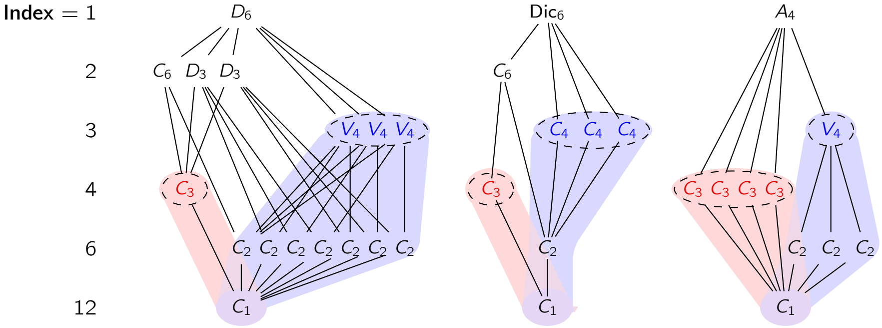
"$\Leftarrow$" Exercise. (Need a result on Sylow p-subgroups.) $\square$

The 4^{th} characterization of nilpotency: normalizers grow

Proposition

A finite group G is nilpotent iff it has no proper fully unnormal subgroups ($H \lneq N_G(H)$).

Since $|\operatorname{cl}_G(H)| = [G : N_G(H)]$, this just means that G has no "maximally wide conjugacy fans."



The 5^{th} and 6^{th} characterizations of nilpotency: Sylow p-subgroups

Proposition

A finite group is nilpotent iff it is the internal direct product of its Sylow p-subgroups.

Proof

"$\Leftarrow$" by previous lemma. ✓

"$\Rightarrow$" Let $P \in \mathrm{Syl}_p(G)$ be a Sylow p-subgroup.

$$\begin{aligned} P &\lneq N_G(P) && \textit{by previous Proposition} \\ &= N_G(N_G(P)) && \textit{property of Sylow p-subgroups} \\ &= G && \textit{contrapositive of Prop.: } N_G(H) = H \textit{ implies } H = G. \end{aligned}$$

Let $P_1, \dots, P_k$ be the distinct Sylow p_i-subgroups of G. We need to verify:

1. $G = P_1 P_2 \cdots P_k$. ✓
2. each $P_i \trianglelefteq G$. ✓
3. each P_i trivially intersects $Q_i := \langle P_j \mid j \neq i \rangle$.

   If $g \in P_i \cap Q_i$, then $|g| = p_i^{\ell}$ divides $\prod_{j \neq i} p_j^{d_j}$, which is co-prime to p_i. ✓

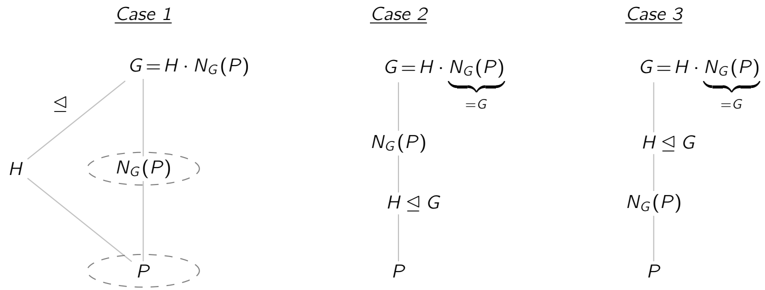
Corollary

A finite group is nilpotent iff all Sylow p-subgroups are normal.

A technical lemma for the 7^{th} characterization of nilpotency.

Frattini argument (exercise)

If G is a finite group and $P \leq H \trianglelefteq G$ with $P \in \mathrm{Syl}_p(H)$, then $G = N_G(P)H$.



Corollary (Case 3)

If G is a finite group and $P \leq H \trianglelefteq G$ with $P \in \mathrm{Syl}_p(H)$ and $N_G(P) \leq H$, then

(i) $H \trianglelefteq G \;\Rightarrow\; P \trianglelefteq G$, (ii) $P \ntrianglelefteq G \;\Rightarrow\; H \ntrianglelefteq G$.

The 7^{th} characterization of nilpotency.

Frattini argument corollary (Case 3)

If G is a finite group and $P \leq H \trianglelefteq G$ with $P \in \mathrm{Syl}_p(H)$ and $N_G(P) \leq H$, then

(i) $H \trianglelefteq G \;\Rightarrow\; P \trianglelefteq G$,

(ii) $P \ntrianglelefteq G \;\Rightarrow\; H \ntrianglelefteq G$.

Proposition

A finite group is nilpotent iff every maximal subgroup is normal.

Proof

"$\Rightarrow$" Let $M \trianglelefteq G$ be maximal normal. Then $M \lneq N_G(M) \leq G \;\Rightarrow\; N_G(M) = G$. ✓

"$\Leftarrow$" Let $P \leq G$ be a Sylow p-subgroup.

Suppose $P \ntrianglelefteq G$, and let M be a maximal subgroup containing its normalizer:

$$P \leq N_G(P) \leq M \lneq G.$$

By Frattini (Case 3), $P \ntrianglelefteq G \;\Rightarrow\; M \ntrianglelefteq G$, a contradiction. $\square$

Summary of nilpotent groups

Theorem

A finite group G is **nilpotent** if any of the following conditions hold:

1. $Z_n = G$ for some n ("*the ascending central series reaches the top*")
2. $L_m = \langle 1 \rangle$ for some m, ("*descending central series reaches the bottom*")
3. G is constructible with central extensions.
4. $H \lneq N_G(H)$ for all proper subgroups, ("*no fully unnormal subgroups*")
5. G is the direct product of its Sylow p-subgroups.
6. All Sylow p-subgroups are normal (or equivalently, $n_p = 1$).
7. Every maximal subgroup of G is normal.