Visual Algebra

Lecture 8.6: Maximal ideals

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Maximal ideals and simple rings

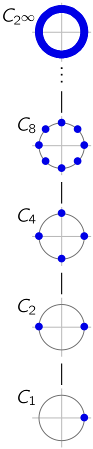
A maximal normal subgroup M of G has no normal subgroups $M \lneq N \lneq G$. Formally:

$$M \leq N \leq G, \quad \text{and} \quad M, N \trianglelefteq G \qquad \Longrightarrow \qquad N = M, \text{ or } N = G.$$

By the correspondence theorem, a normal subgroup $M \trianglelefteq G$ is maximal iff G/M is simple.

The Prüfer group C_{p^∞} of all p^n-th roots of unity ($n \in \mathbb{N}$) has no maximal normal subgroups:

$$\langle 1 \rangle \leq C_p \leq C_{p^2} \leq C_{p^3} \leq \cdots \leq C_{p^\infty}, \qquad C_n = \left\{ e^{2\pi ik/n} \mid k \in \mathbb{N} \right\} \subseteq \mathbb{C}.$$



Definition

An ideal $I \subsetneq R$ is maximal if $I \subseteq J \trianglelefteq R$ implies $J = I$ or $J = R$.

A ring R is simple if its only (two-sided) ideals are 0 and R.

The following is immediate by the correspondence theorem.

Remark

An ideal $M \trianglelefteq R$ is maximal iff R/M is simple.

Maximal ideals and simple rings

Simple rings have no nontrivial proper ideals. Proper ideals cannot contain units.

In a field, *every* nonzero element is a unit. Therefore, fields have no nontrivial proper ideals.

Proposition

A commutative ring R with unity is simple iff it is a field.

Proof

"$\Rightarrow$": Assume R is simple. Then $(a) = R$ for any nonzero $a \in R$.

Thus, $1 \in (a)$, so $1 = ba$ for some $b \in R$, so $a \in U(R)$ and R is a field. ✓

"$\Leftarrow$": Let $I \subseteq R$ be a nonzero ideal of a field R. Take any nonzero $a \in I$.

Then $a^{-1}a \in I$, and so $1 \in I$, which means $I = R$. ✓ □

Theorem

Let R be a commutative ring with 1. The following are equivalent for an ideal $I \subseteq R$.

(i) I is maximal; (ii) R/I is simple; (iii) R/I is a field.

Examples of maximal ideals & simple rings

1. The maximal ideals of $R = \mathbb{Z}$ are $M = (p)$. The quotient field is $\mathbb{Z}/(p) \cong \mathbb{Z}_p$
2. Maximal ideals of $R = \mathbb{Z}[x]$ includes those of the form
$$(x, p) = \{xf(x) + p \cdot g(x) \mid f, g \in \mathbb{Z}[x]\} = \{a_n x^n + \cdots + a_1 x + pa_0 \mid a_i \in \mathbb{Z}\}.$$
In the quotient field, "$x := 0$" and "$p := 0$", and so
$$\mathbb{Z}[x]/(x, p) = \{a_0 + M \mid a_0 = 0, \dots, p-1\} \cong \mathbb{Z}_p.$$
3. Let $R = \mathbb{Q}[x]$. The ideal
$$(x) = \{xf(x) \mid f \in \mathbb{Q}[x]\} = \{a_n x^n + \cdots + a_1 x \mid a_i \in \mathbb{Z}\}$$
is maximal. In the quotient field, "$x := 0$", and so
$$\mathbb{Q}[x]/(x) = \{a_0 + M \mid a_0 \in \mathbb{Q}\} \cong \mathbb{Q}.$$
4. In the multivariate ring $R = \mathbb{F}[x, y]$ over a field, the ideal
$$I = (x, y) = \{x \cdot f(x, y) + y \cdot g(x, y) \mid f, g \in R\}$$
of polynomials with no constant term is maximal. The quotient field is $R/I \cong \mathbb{F}$.
5. Examples of simple noncommutative rings: $\mathbb{H}$, and $\mathrm{Mat}_n(\mathbb{F})$.

Existence of maximal ideals

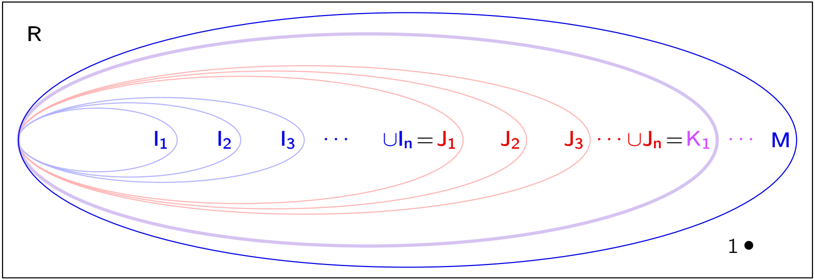
Given an ideal $I_1 \subsetneq R$. Let's try to find a maximal ideal that contains it.

If we have a sequence $I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots$ of ideals, then $J_1 := \bigcup I_k \subsetneq R$ is an ideal.

If this isn't maximal, find $r_2 \notin J_1$, and let $J_2 = (J_1, r_2)$, and repeat this process.

Suppose we have $J_1 \subsetneq J_2 \subsetneq J_3 \subsetneq \cdots$. Then $K_1 := \bigcup J_k \subsetneq R$ is an ideal.

Is this process going to "stop"?



Assuming the axiom of choice: **YES!**

Ordinals and transfiniteness

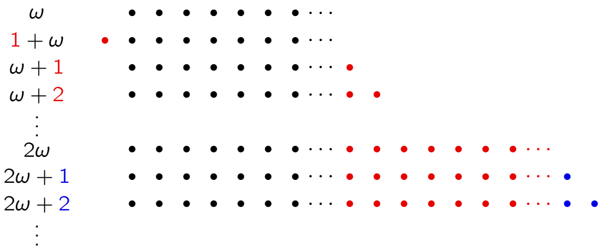
A set is well-ordered if every subset has a minimal element.

The natural numbers $\mathbb{N}$ are well-ordered, the integers $\mathbb{Z}$ are not.

Loosely speaking, an ordinal is an equivalence class of well-ordered sets.

Ordinal arithmetic involves addition, multiplication, and exponentiation.

The ordinal for $\mathbb{N}$ is denoted ω. Some things may be surprising, like $\omega = 1 + \omega \neq \omega + 1$.

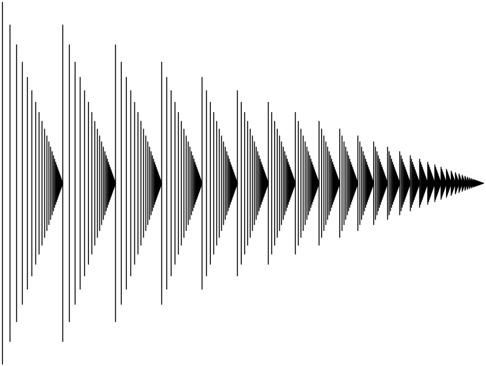


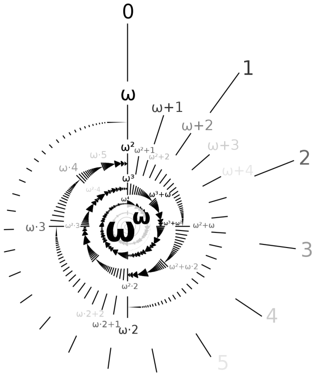
There are three types:

- finite ordinals
- successor ordinals
- limit ordinals

Ordinals and transfiniteness

Here are some depictions of the ordinals ω^2 and ω^ω.





Mathematical induction and recursion is traditionally done over the ordinal ω.

Over general ordinals, these are callled transfinite induction and recursion.

The axiom of choice is needed.

The maximal ideal of $I \subseteq R$ is basically the result of a *transfinite union*.

Existence of maximal ideals

Zorn's lemma (equivalent to the axiom of choice)

If $\mathcal{P} \neq \emptyset$ is a poset in which every chain has an upper bound, then $\mathcal{P}$ has a maximal element.

Proposition

If R is a ring with 1, then every ideal $I \neq R$ is contained in a maximal ideal M.

Proof

Fix I, and let $\mathcal{P}$ be the poset of *proper ideals* containing it.

Every chain $I \subseteq I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ has an upper bound, $\bigcup I_k \subsetneq R$.

Zorn's lemma guarantees a maximal element M in $\mathcal{P}$, which is a maximal ideal containing I.

Corollary

If R is a ring with 1, then every non-unit is contained in a maximal ideal M.

Do you see why this doesn't work for maximal subgroups?