Visual Algebra

Lecture 8.9: Radical ideals

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Radical ideals

Loosely speaking, a radical I of R is an ideal of "bad elements;" the quotient R/I is "nice."

Preview example 1

The nilradical of R has two equivalent characterizations:

- The set of nilpotent elements.
- The intesection of nonzero prime ideals.

$$\mathfrak{N}(R) := \big\{x \in R \mid x^n = 0 \text{ for some } n \in \mathbb{N}\big\} = \bigcap_{0 \neq P \subsetneq R \text{ prime}} P.$$

The quotient $R/\mathfrak{N}(R)$ is a "subdirect product" of integral domains.

Preview example 2

The Jacobson radical of R has two equivalent characterizations:

- The set of elements $x \in R$ that annihilate simple R-modules, i.e., $xM = 0$ for all M.
- The intesection of maximal ideals.

$$\operatorname{Jac}(R) := \big\{x \in R \mid 1 - rx \text{ is a unit for all } r \in R\big\} = \bigcap_{M \subsetneq R \text{ max'l}} M.$$

The quotient $R/\operatorname{Jac}(R)$ is a "subdirect product" of fields.

Subdirect products

Think of a subdirect product as being "*almost a direct product*."

The "diagonal subring" $S = \{(n,n) \mid n \in \mathbb{Z}\} \subseteq \mathbb{Z} \times \mathbb{Z}$ is a subdirect product because:

(i) It is a subring of $\mathbb{Z} \times \mathbb{Z}$.

(ii) It projects onto each component of the product.

Let $\{R_i \mid i \in I\}$ be a family of rings with direct product and projection maps

$$R = \prod_{i \in I} R_i, \qquad \pi_j \colon R \longrightarrow R_j$$
$$(r_i)_{i \in I} \longmapsto r_j.$$

Definition

A ring S is a subdirect product of R if there is $\iota \colon S \hookrightarrow R$ such that each composition

$$S \xrightarrow{\iota} R \xrightarrow{\pi_j} R_j, \qquad s \xmapsto{\iota} (r_i)_{i \in I} \xmapsto{\pi_j} r_j$$

is surjective.

Subdirect products can be defined analogously for sets, groups, vector spaces, etc.

Subdirect products

Proposition

Let $\{J_i \mid i \in I\}$ be a family of ideals of R with $J = \bigcap_{i\in I} J_i$. Then R/J is a subdirect product of $\{R/J_i \mid i \in I\}$.

Proof

The map

$$\phi\colon R \longrightarrow \prod_{i\in I} R/J_i, \qquad x \longmapsto (x + J_i)_{i\in I}$$

is a homomorphism with $\mathrm{Ker}(\phi) = J$. By the FHT for rings, there is an isomorphism

$$\iota\colon R/J \longrightarrow \mathrm{Im}(\phi) \leq \prod_{i\in I} R/J_i.$$

The composition of maps is surjective, for each $j \in I$:

$$R/J \stackrel{\iota}{\hookrightarrow} \prod_{i\in I} R/J_i \stackrel{\pi_j}{\longrightarrow} R/J_j, \qquad r + J \stackrel{\iota}{\longmapsto} \prod_{i\in I}(r + J_i)_{i\in I} \stackrel{\pi_j}{\longmapsto} r + J_j.$$

The nilradical of a ring

Definition (membership test)

The nilradical of R is the set of nilpotent elements:

$$\mathfrak{N}(R) = \{a \in R \mid a^n = 0, \text{ for some } n \in \mathbb{N}\}.$$

Proposition

$\mathfrak{N}(R)$ is an ideal of R.

Proof

Subgroup: Suppose $x, y \in \mathfrak{N}(R)$, and $x^n = y^m = 0$. Using the binomial theorem,

$$(x-y)^{n+m} = \sum_{i=1}^{n+m} a_i x^i y^{n+m-i}.$$

Either $i \geq n$ (so $x^i = 0$) or $n+m-i \geq m$ (so $y^{n+m-i} = 0$) must hold. ✓

Ideal: If $x^n = 0$ and $r \in R$, then $(rx)^n = r^n x^n = 0$, so $rx \in \mathfrak{N}(R)$. ✓

The nilradical of a ring

Proposition (ideal characterization)

The nilradical is the intersection of all nonzero prime ideals: $\mathfrak{N}(R) = \displaystyle\bigcap_{P \subsetneq R \text{ prime}} P$.

Proof

"$\subseteq$" Let $a \in \mathfrak{N}(R)$ and $P \subseteq R$ prime. Let $n \geq 1$ be minimal such that $a^n \in P$.

Since $a^{n-1}a \in P$ (prime), either $a^{n-1} \in P$ (contradiction) or $a \in P$. Thus $a \in \cap P$. ✓

"$\supseteq$" Suppose $a \notin \mathfrak{N}(R)$; we'll show $a \notin \cap P$.

$$\mathcal{S} = \big\{ J \trianglelefteq R \text{ s.t. } a^n \notin J \text{ for all } n \in \mathbb{N} \big\}.$$

$\mathcal{S}$ is nonempty since it contains (0).

We can apply Zorn's lemma (why?) to get a maximal element $P \in \mathcal{S}$.

P **is prime**: Say $xy \in P$ but $x, y \notin P$. Then $a^n \in \underbrace{(x) + P}_{\notin \mathcal{S}}$ and $a^m \in \underbrace{(y) + P}_{\notin \mathcal{S}}$ for some n, m.

But then $a^{nm} \in \underbrace{(xy) + P}_{= P}$, contradicting the fact that $P \in \mathcal{S}$. □

The Jacobson radical of a ring

Definition (membership test)

The Jacobson radical of R is the set

$$\mathbf{Jac}(R) = \big\{x \in R \mid 1 - rx \text{ is a unit for all } r \in R\big\}.$$

Proposition (ideal characterization)

The Jacobson radical is the intersection of all maximal ideals: $\mathbf{Jac}(R) = \displaystyle\bigcap_{M \subsetneq R \text{ prime}} M.$

Proof

"$\subseteq$": Suppose $1 - rx \notin U(R)$ for some $x \in R$, and let M be a maximal ideal containing it.

If $r \in \mathbf{Jac}(R)$, then $r \in M$, which is impossible because

$$1 = \underbrace{(1 - rx)}_{\in M} + \underbrace{rx}_{\in M} \in M.$$

The Jacobson radical of a ring

Definition (membership test)

The Jacobson radical of R is the set

$$\mathrm{Jac}(R) = \big\{x \in R \mid 1 - rx \text{ is a unit for all } r \in R\big\}.$$

Proposition (ideal characterization)

The Jacobson radical is the intersection of all maximal ideals: $\mathrm{Jac}(R) = \displaystyle\bigcap_{M \subsetneq R \text{ max'l}} M.$

Proof

"$\supseteq$": Suppose $x \not\in M$ for some maximal ideal M. Then

$$R = M + (x) = \big\{m + rx \mid m \in M,\ r \in R\big\},$$

so we can write

$$1 = m + rx \qquad \Longrightarrow \qquad \underbrace{1 - xy}_{\not\in U(R)} = m \in M.$$

Quotients by radicals are subdirect products

Corollary

The quotient $R/\mathfrak{N}(R)$ is a subdirect product of integral domains.

Proof

Let $\{P_i \mid i \in I\}$ be the set of prime ideals of R; recall $\mathfrak{N}(R) = \bigcap_{i \in I} P_i$.

Then $R/\mathfrak{N}(R)$ is a subdirect product of $\{R/P_i \mid i \in I\}$, which are all integral domains. $\square$

Corollary

The quotient $R/\operatorname{\mathbf{Jac}}(R)$ is a subdirect product of fields.

Proof

Let $\{M_i \mid i \in I\}$ be the set of maximal ideals of R; recall $\operatorname{\mathbf{Jac}}(R) = \bigcap_{i \in I} M_i$.

Then $R/\operatorname{\mathbf{Jac}}(R)$ is a subdirect product of $\{R/M_i \mid i \in I\}$, which are all fields. $\square$

The radical of an ideal

Definition

The radical of an ideal I is the set

$$\sqrt{I} := \{ r \in R \mid r^n \in I, \text{ for some } n \in \mathbb{N} \}.$$

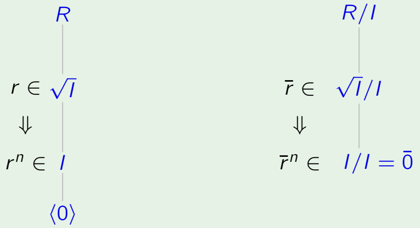
If $\sqrt{I} = I$, then I is a radical ideal.

The nilradical is just the radical of the zero ideal: $\mathfrak{N}(R) = \sqrt{0}$.

Proposition

$\mathfrak{N}(R/I) = \sqrt{I}/I$.

Proof (sketch; details for HW)



The radicals of an ideal

Definition

The Jacobson radical of I is the intersection of all maximal ideals that contain it:

$$\mathrm{jac}(I) := \bigcap_{I \subseteq M \trianglelefteq R} M.$$

The Jacobson radical of R is the Jacobson radical of the zero ideal: $\mathrm{Jac}(R) := \mathrm{jac}(0)$.

Definition / proposition

The radical of I is the intersection of all prime ideals that contain it:

$$\sqrt{I} = \bigcap_{I \subseteq P \trianglelefteq R} P.$$

The nilradical of R is the radical of the zero ideal: $\mathfrak{N}(R) := \sqrt{0}$.

Proposition (HW)

In a commutative ring with 1, an ideal P is prime iff it is primary and radical.