

Visual Algebra

Lecture 8.11: Rings of fractions

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Motivation: constructing \mathbb{Q} from \mathbb{Z}

Rational numbers are ordered pairs under an equivalence, e.g., $\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \dots$

Equivalence of fractions

Given $a, b, c, d \in \mathbb{Z}$, with $b, d \neq 0$,

$$\frac{a}{b} = \frac{c}{d} \quad \text{if and only if} \quad ad = bc.$$

We can mimic this construction in any integral domain.

Definition

Given an integral domain R , its **field of fractions** is the set

$$R \times R^* = \{(a, b) \mid a, b \in R, b \neq 0\},$$

under the **equivalence** $(a_1, b_1) \sim (a_2, b_2)$ iff $a_1 b_2 = b_2 a_1$.

Denote the class containing (a, b) as a/b . Addition and multiplication are defined as

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \quad \text{and} \quad \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

It's not hard to show that $+$ and \times are **well-defined**.

Embedding integral domains in fields

Lemma

In the construction of the field of fractions from R , we must verify:

- \sim is an equivalence relation
- the $+$ and \times operations are well-defined on $(R \times R^*)/\sim$
- the additive identity is $0/r$ for any $r \in R^*$
- the multiplicative identity is r/r for any $r \in R^*$
- $(a, b)^{-1} = b/a$.

Integral domain	Field of fractions
\mathbb{Z} (integers)	\mathbb{Q} (rationals)
$\mathbb{Z}[i]$ (Gaussian integers)	$\mathbb{Q}(i)$ (Gaussian rationals)
$F[x]$ (polynomials)	$F(x)$ (rational functions)

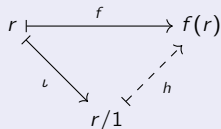
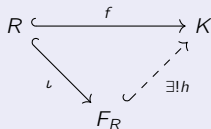
Every integral domain canonically embeds into its field of fractions, via $r \mapsto r/1$.

Moreover, this is the *minimal* field containing R .

Co-universal property of the field of fractions

Proposition

Let R be an integral domain with embedding $\iota: R \hookrightarrow F_R$ into its field of fractions. Then for every other embedding $f: R \hookrightarrow K$ into a field, there is a unique $h: F_R \hookrightarrow K$ such that $h \circ \iota = f$.



Proof

Define the map

$$h: F_R \longrightarrow K, \quad h(a/b) \longmapsto h(a/1)h(b/1)^{-1} = f(a)f(b)^{-1}.$$

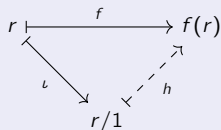
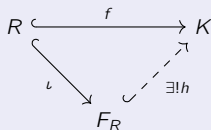
We need to show that h is

- (i) well-defined
- (ii) a ring homomorphism,
- (iii) injective
- (iv) unique.

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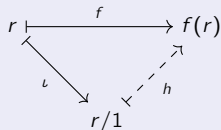
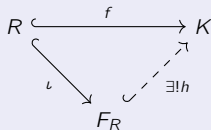
(i) **Well-defined.** Suppose $a/b = c/d \Leftrightarrow ad = bc \Leftrightarrow ab^{-1} = cd^{-1}$.

$$h(a/b) = f(a)f(b^{-1}) = f(ab^{-1}) = f(cd^{-1}) = f(c)f(d^{-1}) = h(c/d). \quad \checkmark$$

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(ii) **Ring homomorphism.** Suppose $a/b = c/d$. Then

$$\begin{aligned} h(a/b \cdot c/d) &= h(ac/bd) = f(ac)f(d^{-1}b^{-1}) = f(a)f(c)f(d^{-1})f(b^{-1}) \\ &= f(a)f(b^{-1}) \cdot f(c)f(d^{-1}) = h(a/b)h(c/d). \end{aligned}$$

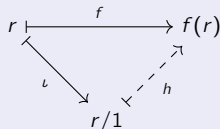
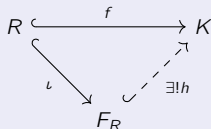
✓

Verification of $h(a/b + c/d) = h(a/b) + h(c/d)$ is similar. (Exercise)

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(iii) **Injective.** It suffices to show that $\text{Ker}(h) = \{0\}$. Suppose

$$0 = h(a/b) = f(a)f(b)^{-1} \in K.$$

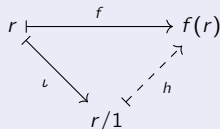
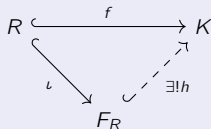
However, $f(b)^{-1} \neq 0$ since f is an embedding and $b \neq 0$.

Thus $f(a) = 0$, so $a = 0$ in R . Thus $a/1 = 0/1$, the zero element in F_R . ✓

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(iv) Uniqueness. Suppose there is another $g: F_R \rightarrow K$ such that $f = g \circ \iota$. Then

$$g(a/b) = g((a/1) \cdot (b/1)^{-1}) = g(a/1)g(b/1)^{-1} = g(\iota(a))g(\iota(b))^{-1} = f(a)f(b)^{-1} = h(a/b),$$

which completes the proof. \square

Rings of fractions and localization

The co-universal property can be used as the *definition* of the field of fractions, allowing:

- the generalization to rings without 1, e.g., $R = 2\mathbb{Z}$. (Exercise: show that $F_{2\mathbb{Z}} = \mathbb{Q}$.)
- the generalization to constructing fractions of certain subsets.

Let R be commutative, $D \subseteq R$ nonempty and **multiplicatively closed** with no zero divisors.

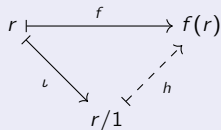
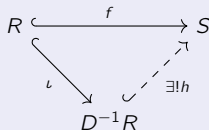
We can carry out the same construction of the set

$$R \times D = \{(r, d) \mid r \in R, d \in D\}, \quad (r_1, d_1) \sim (r_2, d_2) \text{ iff } r_1 d_2 = r_2 d_1.$$

The resulting ring is the **localization of R at D** , denoted $D^{-1}R$.

Proposition (HW)

Let R be a commutative ring with embedding $\iota: R \hookrightarrow D^{-1}R$. Then for every other embedding $f: R \hookrightarrow S$ to a ring where $f(D)$ are units, there is a unique $h: D^{-1}R \hookrightarrow S$ such that $h \circ \iota = f$.



Localization with zero divisors

We can generalize this further! Allow D to contain zero divisors.

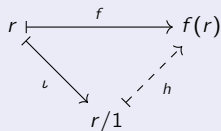
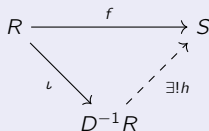
The mapping $R \rightarrow D^{-1}R$ sending r to its equivalence class is no longer injective:

$$\iota: R \longrightarrow D^{-1}R, \quad \iota(z) = 0, \quad \text{for all zero divisors } z \in D.$$

We still have a co-universal property, that could have been the definition.

Proposition (exercise)

Let R be a commutative ring with $\iota: R \rightarrow D^{-1}R$. For every other $f: R \rightarrow S$ to a ring where the non zero-divisors in $f(D)$ are units, there is a unique $h: D^{-1}R \rightarrow S$ such that $h \circ \iota = f$.



Thus, $D^{-1}R$ is the “smallest ring” where all non zero-divisors in D are invertible.

Examples of rings of fractions

1. If R is an integral domain and $D = R^*$, then $D^{-1}R$ is its **field of fractions**.
2. If D is the set of non zero divisors, then $D^{-1}R$ is the **total ring of fractions** of R .
3. If non-unit of R is a zero divisor, then R is equal to its total ring of fractions.

Examples include $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$.

In these rings, every prime ideal is maximal (exercise).

4. The localization of $R = F[x]$ at $D = \{x^n \mid n \in \mathbb{Z}\}$ are the **Laurent polynomials**:

$$D^{-1}R = F[x, x^{-1}] = \{a_{-m}x^{-m} + \cdots + a_{-1}x^{-1} + a_0 + a_1x + \cdots + a_nx^n \mid a_i \in F\}.$$

5. If $R = \mathbb{Z}$ and $D = \{5^n \mid n \in \mathbb{N}\}$, then

$$D^{-1}R = \mathbb{Z}[\frac{1}{5}] = \left\{a_0 + \frac{a_1}{5} + \frac{a_2}{5^2} + \cdots + \frac{a_n}{5^n} \mid a_i \in \mathbb{Z}\right\}.$$

which are “*polynomials in $\frac{1}{5}$* ” over \mathbb{Z} .

6. If $D = R - P$ for a prime ideal, then $R_P := D^{-1}R$ is the **localization of R at P** . It is a **local ring** – it has a unique maximal ideal, PR_P .