Logarithm and Exponential Functions

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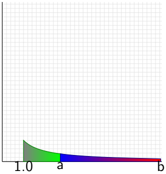
Outline

Let's look at addition of logarithms in more general terms. What happens if we add two areas under the curve $\frac{1}{x}$? Let's assume we have two real numbers a and b, both of which are larger than 1. For convenience, we can assume also that b is larger than a. Hence, we have the picture shown below.



The graph of the curve $\frac{1}{x}$ is shown. We seek a point c on the x - axis where the

$$\int_1^a \frac{1}{t} dt + \int_1^b \frac{1}{t} dt \; = \; \int_1^c \frac{1}{t} dt$$

Figure: We consider two different areas under the curve: the area from 1 to a and the area from 1 to b with a and b both larger than 1

And we have that

$$\int_1^a \frac{1}{x} \, dx \; < \; \int_1^b \frac{1}{x} \, dx.$$

Now make a change of variable in the first area integral. We let $u = bx$ and we find

$$\int_1^a \frac{1}{x} \, dx \; = \; \int_b^{ab} \frac{1}{u} \, du$$

Since the name of the variable of integration is not important, when we add the two areas, we obtain

$$\int_1^a \frac{1}{x} \, dx + \int_1^b \frac{1}{x} \, dx \; = \; \int_b^{ab} \frac{1}{x} \, dx + \int_1^b \frac{1}{x} \, dx = \int_1^{ab} \frac{1}{x} \, dx$$

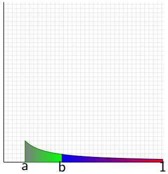
We conclude that

$$\int_1^a \frac{1}{x}\,dx + \int_1^b \frac{1}{x}\,dx \;=\; \int_1^{ab} \frac{1}{x}\,dx$$

We can argue in a similar way if the relationship between a and b is reversed so that a is larger than b. Finally, if a and b are the same number, the argument is still valid, although easier!

We have shown how the argument goes for the case that both a and b are larger than 1.

It is very similar to argue the case for a and b both less than 1. The picture is now what you see below. We will assume that a is less than b just as before.



The graph of the curve $\frac{1}{x}$ is shown. We seek a point c on the x - axis where the

$$\int_a^1 \frac{1}{t}dt + \int_b^1 \frac{1}{t}dt \;=\; \int_c^1 \frac{1}{t}dt$$

Figure: We consider two different areas under the curve: the area from a to 1 and the area from b to 1 with a and b both less than 1

Since a is less than b, this time we have

$$\int_a^1 \frac{1}{x}\,dx \quad > \quad \int_b^1 \frac{1}{x}\,dx$$

Now make a change of variable in the second area integral. We let $u = ax$ and we find

$$\int_b^1 \frac{1}{x}\,dx \quad = \quad \int_{ab}^a \frac{1}{u}\,du$$

Since $0 < a < 1$, we see that

$$ab - a \quad = \quad b\,(a - 1) < 0.$$

Thus, we see the numbers ab, a and b are ranked as $ab < a < b < 1$. Then, since the name of the variable of integration is not important, when we add the two areas, we obtain

$$\int_a^1 \frac{1}{x}\,dx + \int_b^1 \frac{1}{x}\,dx \quad = \quad \int_a^1 \frac{1}{x}\,dx + \int_{ab}^a \frac{1}{x}\,dx = \int_{ab}^1 \frac{1}{x}\,dx$$

We conclude that

$$\int_a^1 \frac{1}{x}\,dx + \int_b^1 \frac{1}{x}\,dx \quad = \quad \int_{ab}^1 \frac{1}{x}\,dx$$

Since a and b are both less than one, the product ab is also less than one. Hence the usual properties of the Riemann Integral then allow us to rewrite as

$$-\int_1^a \frac{1}{x}\,dx - \int_1^b \frac{1}{x}\,dx \quad = \quad -\int_1^{ab} \frac{1}{x}\,dx$$

Multiplying through by -1, we have

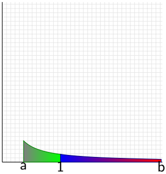
$$\int_1^a \frac{1}{x}\,dx + \int_1^b \frac{1}{x}\,dx \quad = \quad \int_1^{ab} \frac{1}{x}\,dx$$

We see in this case, c is ab.

The last case is where a is less than 1 and b is larger than one. We want to find a number c so that

$$\int_1^a \frac{1}{x}\,dx + \int_1^b \frac{1}{x}\,dx \quad = \quad \int_1^c \frac{1}{x}\,dx$$

The difference in this case is that the first integral $\int_1^a \frac{1}{x}\,dx$ is negative. This case is shown in Figure 3.



The graph of the curve $\frac{1}{x}$ is shown. We seek a point c on the x - axis where the

$$\int_a^1 \frac{1}{t} dt + \int_1^b \frac{1}{t} dt \quad = \quad \int_1^c \frac{1}{t} dt$$

Figure: We consider two different areas under the curve: the area from a to 1 and the area from 1 to b with a less than 1 and b larger than 1

Thus we want to find a c so that

$$-\int_a^1 \frac{1}{x}\,dx + \int_1^b \frac{1}{x}\,dx \quad = \quad \int_1^c \frac{1}{x}\,dx$$

Now make a change of variable in $-\int_a^1 \frac{1}{x}\,dx$. We let $u = bx$ and we find

$$-\int_a^1 \frac{1}{x}\,dx \quad = \quad -\int_{ab}^b \frac{1}{u}\,du$$

Since the name of the variable of integration is not important, we find

$$\begin{aligned} \int_1^a \frac{1}{x}\,dx + \int_1^b \frac{1}{x}\,dx \quad &= \quad -\int_{ab}^b \frac{1}{x}\,dx + \int_1^b \frac{1}{x}\,dx \\ &= \quad \int_b^{ab} \frac{1}{x}\,dx + \int_1^b \frac{1}{x}\,dx = \int_1^{ab} \frac{1}{x}\,dx \end{aligned}$$

We conclude that

$$\int_1^a \frac{1}{x}\,dx + \int_1^b \frac{1}{x}\,dx \quad = \quad \int_1^{ab} \frac{1}{x}\,dx.$$

Thus, in the case a is less than 1 and b is larger than 1, c is ab also.

We can summarize all these results as follows

Theorem

Sum of Logs Rule:
If a and b are two positive numbers, then $\ln(a) + \ln(b) = \ln(ab)$.

Now note that if $a > 0$, then applying the rule above, we have

$$\ln(a/a) = \ln(a\ (1/a)) \quad = \quad 0 \Longrightarrow \ln(a) + \ln(1/a) = 0$$

Thus, we have $\ln(1/a) = -\ln(a)$ for all positive a. This tells us immediately that if a and b are positive, we have

$$\ln(a/b) \quad = \quad \ln(a) - \ln(b)$$

which leads to the theorem

Theorem

Difference of Logs Rule:
If a and b are two positive numbers, then $\ln(a) - \ln(b) = \ln(a/b)$.

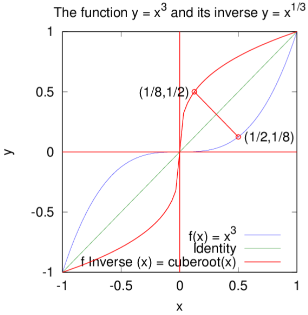
- Let's backup and talk about the idea of an inverse function. Say we have a function $y = f(x)$ like $y = x^3$. Take the cube root of each side to get $x = y^{1/3}$. Just for fun, let $g(x) = x^{1/3}$; i.e., we switched the role of x and y in the equation $x = y^{1/3}$.
- Now note some interesting things:

$$\begin{aligned} f(g(x)) &= f(x^{1/3}) = \left(x^{1/3}\right)^3 = x \\ g(f(x)) &= g(x^3) = \left(x^3\right)^{1/3} = x. \end{aligned}$$

- Now the function $I(x) = x$ is called the identity because it takes an x as an input and does nothing to it. The output value is still x. So we have $f(g) = I$ and $g(f) = I$. When this happens, the function g is called the **inverse** of f and is denoted by the special symbol f^{-1}.
- Of course, the same is true going the other way: f is the inverse of g and could be denoted by g^{-1}. Another more abstract way of saying this is

$$f^{-1}(x) = y \quad \Leftrightarrow \quad f(y) = x.$$

- Now look at next picture We draw the function x^3 and its inverse $x^{1/3}$ in the unit square. We also draw the identity there which is just the graph of $y = x$; i.e. a line with slope 1.
- If you take the point $(1/2, 1/8)$ on the graph of x^3 and draw a line from it to the inverse point $(1/8, 1/2)$ you'll note that this line is perpendicular to the line of the identity. This will always be true with a graph of a function and its inverse.
- Now also note that x^3 has a positive derivative always and so is always increasing. It seems reasonable that if we had a function whose derivative was positive all the time, we could do this same thing. We could take a point on that function's graph, say (c, d), reverse the coordinates to (d, c) and the line connecting those two pairs would be perpendicular to the identity line just like in our figure. So we have a geometric procedure to define the inverse of any function that is always increasing.



- What about $\ln(x)$? It has derivative $1/x$ for all positive x, so it must be always increasing. So it has a differentiable inverse which we can call $\ln^{-1}(x)$ which is called the **exponential function** which is denoted by $\mathbf{exp}(x)$.
- The inverse is defined by the if and only relationship $(\ln)^{-1}(x) = y \Leftrightarrow \ln(y) = x$
- or, using the exp notation $\exp(x) = y \Leftrightarrow \ln(y) = x$.
- A little thought tells us the range of $\ln(x)$ is all real numbers as for $x > 1$, $\ln(x)$ gets as large as we want and for $0 < x < 1$, as x gets closer to zero, the negative area $-\int_x^1 1/t dt$ approaches $-\infty$.
- By definition then
  - $\ln(\exp(x)) = x$ for $-\infty < x < \infty$; ie for all x.
  - $\exp(\ln(x)) = x$ for all $x > 0$.

- We know $\ln(\exp(x)) = x$. Take the derivative of both sides:

$$\Big(\ln(\exp(x)\Big)' = \Big(x\Big)' = 1$$

- Using the chain rule, for any function $u(x)$,

$$\Big(\ln(u(x)\Big)' = \frac{1}{u(x)}\, u'(x).$$

- So

$$\Big(\ln(\exp(x)\Big)' = \frac{1}{\exp(x)}\Big(\exp(x)\Big)'.$$

- Using this, we see $\frac{1}{\exp(x)}\Big(\exp(x)\Big)' = 1$ and so

$$\Big(\exp(x)\Big)' = \exp(x).$$

It is easy to see by a simple POMI that $\ln(a^p) = p\ln(a)$ when p is a positive integer. So, if $u = a^{p/q}$, we have $u^q = a^{1/p}$.

We also know by another simple POMI that for any positive integer q, $\ln(a^1) = \ln((a^{1/q})^q) = q\ln(a^{1/q})$. Thus, $\ln(a^{1/q}) = (1/q)\ln(a)$.

Combining, we see

$$\ln(a^{p/q}) = p\ln(a^{1/q}) = (p/q)\ln(a)$$

We can do this for any $p/q \in \mathbb{Q}$ even if p or q are negative integers.

Now let (x_n) be any sequence of real numbers which converges to the number x. By the continuity of exp we then know

$$\lim_{n\to\infty} \exp(x_n \ln(a)) = \exp((\lim_{n\to\infty} x_n)\ln(a))$$

Thus, we can uniquely define the function a^x by $a^x = \lim_{n\to\infty} \exp(x_n \ln(a))$.

In particular if $x_n \to \alpha$, we have $a^\alpha = \exp(\alpha \ln(a))$ is a uniquely defined continuous function which by the chain rule is differentiable.

Also, this means the function $e^x = \exp(x \ln(e)) = \exp(x)$ is another way to write our inverse function $\exp(x)$. Now let α be any real number and let (p_n/q_n) be a sequence of rational numbers which converges to α. We can always find such a sequence because not matter what circle $B_r(\alpha)$ we choose, we can find a rational number within a distance of r from α. We also know $\ln(x)$ is continuous on $(0, \infty)$. Then for a positive number a, we have

$$\lim_{n\to\infty} \ln(a^{p_n/q_n}) \;=\; \lim_{n\to\infty} (p_n/q_n) \ln(a) = \alpha \ln(a)$$

But the function a^x is continuous, so

$$\lim_{n\to\infty} (a^{p_n/q_n}) \;=\; a^\alpha \Longrightarrow \ln(a^\alpha) = \alpha \ln(a)$$

This is called the power rule for the logarithm function: $\ln(a^r) = r \ln(a)$ for all positive a and any real number r.

Theorem

Properties Of The Natural Logarithm
The natural logarithm of the real number x satisfies

- $\ln$ *is a continuous function of x for positive x,*
- $\lim_{x\,\to\,\infty}\ \ln(x) \;=\; \infty$,
- $\lim_{x\,\to\,0^+}\ \ln(x) \;=\; -\infty$,
- $\ln(1) \;=\; 0,\ \ln(e) \;=\; 1$,
- $(\ \ln(x)\ )' \;=\; \frac{1}{x}$,
- *If x and y are positive numbers then* $\ln(xy) = \ln(x) \,+\, \ln(y)$.
- *If x and y are positive numbers then* $\ln\left(\frac{x}{y}\right) = \ln(x) \,-\, \ln(y)$.
- *If x is a positive number and y is any real number then* $\ln\left(x^y\right) = y\ \ln(x)$.

Let $u = \ln(x)$ and $v = \ln(y)$. Then

- $x = exp(u)$ and $y = exp(v)$.
- Let $w = \exp(u + v)$. Then by definition, $\ln(w) = u + v$.
- But $u + v = \ln(x) + \ln(y) = \ln(x\,y)$.
- So $\ln(w) = \ln(x\,y)$.
- Then by definition, $w = x\,y = exp(u)\,exp(v)$.
- So in general $exp(u + v) = \exp(u)\,exp(v)$.

Let $u = \ln(x)$ and $v = \ln(y)$. Then

- $x = exp(u)$ and $y = exp(v)$.
- Let $w = \exp(u - v)$. Then by definition, $\ln(w) = u - v$.
- But $u - v = \ln(x) - \ln(y) = \ln(x/y)$.
- So $\ln(w) = \ln(x/y)$.
- Then by definition, $w = x/y = exp(u)/exp(v)$.
- Further, $exp(0) = 1$ because $\ln(1) = 0$, so $\exp(x - x) = \exp(0) = 1$ implying $\exp(x)\ \exp(-x) = 1$. Hence, $\exp(-x) = 1/exp(x)$.
- So in general $exp(u - v) = \exp(u)/exp(v) = \exp(u)\,exp(-v)$.

Let $u = \ln(x)$ and r be any power. Then

- $x = exp(u)$.
- Let $w = (\exp(u))^r$. Then by definition, $\ln(w) = r\ \ln(\exp(u)) = r\ u$.
- Or $w = exp(r\ u)$.
- So in general $\Big(exp(u) \Big)^r = \exp(ru)$.

Theorem

Properties Of The Exponential Function
The exponential function of the real number x, $\exp(x)$, satisfies

- $\exp$ *is a continuous function of x for all x,*
- $\lim_{x \to \infty} \exp(x) = \infty$,
- $\lim_{x \to -\infty} \exp(x) = 0$,
- $\exp(0) = 1$,
- $(\exp(x))' = \exp(x)$,
- *If x and y are any numbers then* $\exp(x + y) = \exp(x)\ \exp(y)$.
- *If x and y are any numbers then* $\exp(x - y) = \exp(x)\ \exp(-y)$ *or* $\exp(x - y) = \frac{\exp(x)}{\exp(y)}$.
- *If x and y are any numbers then* $\Big(\exp(x)\Big)^y = \exp(xy)$.

Another way to derive the logarithm and exponential functions is to define $e = \lim_{n \to \infty} (1 + 1/n)^n$. Then we derive the properties of e^x and show it has an inverse which we call $\ln(x)$.

Let's assume our new way of doing this via integration gives the logarithm type function $H(x) = \int_1^x 1/t dt$ with inverse $F(x)$. The functions $F(x)$ and e^x have the same Taylor polynomials and error functions. We have

$$
\begin{aligned}
F(x) &= \sum_{n=0}^{N} x^n/n! + F(c_{N+1})\, x^{N+1}/(N+1)! \\
e^x &= \sum_{n=0}^{N} x^n/n! + e^{d_{N+1}}\, x^{N+1}/(N+1)!
\end{aligned}
$$

where c_{N+1} and d_{N+1} are points between 0 and x. But all these derivatives of any order are still $F(x)$ and e^x. Thus

$$
F(x) - e^x = \left(F(c_{N+1}) + e^{d_{N+1}}\right) \frac{x^{N+1}}{(N+1)!}
$$

Now both F and exp are continuous on the interval from 0 to x, so there are constants A_x and B_x which are bounds for them. So we have

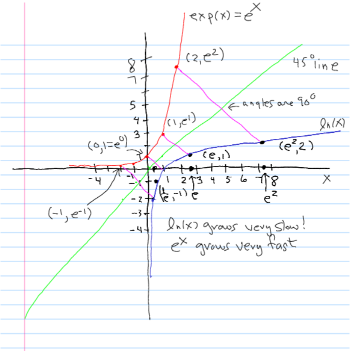
$$
|F(x) - e^x| \leq (A_x + B_x) \frac{x^{N+1}}{(N+1)!} \to 0
$$

as $N \to \infty$ since x is fixed.
Thus, given $\epsilon > 0$, there is a Q so that $(A_x + B_x) \frac{|x|^{N+1}}{(N+1)!} < \epsilon$ if $N + 1 > Q$. Hence, $|F(x) - e^x| < \epsilon$ with ϵ arbitrary. This tells us $F(x) = e^x$.

Another easier way to see this (!!) is to note since $H'(x) = (\ln(x))' = 1/x$, $H(x)$ and $\ln(x)$ are both antiderivatives of $1/x$. Hence, they differ by a constant. But since $H(1) = \ln(1) = 0$, they must be the same.

So $e = f$ even though we have used two different ways to find them!

Let's graph exp(x) and ln(x) on the same graph.



Homework 15

15.1 Prove $e^{-t} \geq 1 - t$ for $0 \leq t \leq 1$.

15.2

(a): Prove $(1-t)^N \geq 1 - Nt$ for $N \geq 1$ for $0 \leq t \leq 1$
(b): Prove $e^{-Nt} \geq 1 - Nt$ for $N \geq 1$ for $0 \leq t \leq 1$.

15.3 Assume f is continuous and satisfies $|f(t)| \leq e^{-\alpha t}$ for $t \geq 0$ and some $\alpha > 0$. Define the sequence (L_n) by $L_n = \int_0^n e^{-st} f(t)dt$ for a fixed s. Prove (L_n) is a Cauchy Sequence of real numbers and hence it must converge when $s + \alpha > 0$.